Chapters 1.6-1.7: Proofs

Wednesday, June 24

Deductive Reasoning

If Trevor gets stuck in traffic he will be late to work. If he is late to work he will be fired. He will not get stuck in traffic if *and only if* he takes the shortcut.
Which of the following conclusions are logically valid? Prove the ones that are.

1. If Trevor does not take the shortcut then he will be fired.
2. If Trevor takes the shortcut then he will not be fired.
3. If Trevor is not late to work then he took the shortcut.

If we lose the game, we will either cry or go out for ice cream (maybe both). Which conclusions are logically valid? Prove the ones that are.

1. If we do not cry and do not go out for ice cream, then we did not lose the game.
2. If we lose the game and do not go out for ice cream, then we will cry.
3. If we do not lose the game and we go out for ice cream, then we will not cry.

Variations on a Theorem

Prove:

1. Suppose $a > b$. If $c > 0$ then $ac > bc$. (There is nothing to prove here–this is an axiom!)
2. Suppose $a > b$. If $ac \leq bc$ then $c \leq 0$.
3. Suppose $a > b$. Then $c \leq 0$ or $ac > bc$.
4. If $c > 0$ and $a \leq b$, then $ac \leq bc$.
5. If $ac \leq bc$ and $c > 0$, then $a \leq b$.

Direct Proofs

Note: Many of these exercises are taken from *How To Prove It*, by Daniel Velleman.
Prove:

1. If $a = 2$ then $a^2 + 2a - 8 = 0$.
2. If $a^2 - 1 = 0$ then $a = 1$ or $a = -1$.
3. If $a^2 = b^2$ then $a = b$ or $a = -b$.
4. If $x^2 = 4$ and $x > 0$ then $x = 2$
5. If $x^2 + y = -3$ and $2x - y = 2$ then $x = -1$.
6. Suppose $3x + 2y \leq 5$. If $x > 1$, then $y < 1$.
7. $3a + 5 = 20$ if and only if $a = 5$.
8. If $a < b$ then $\frac{a+b}{2} < b$.
9. Prove that the quadratic formula is correct.
10. The product of two odd numbers is odd.
11. The sum of an even number and an odd number is odd.
12. The product of two even numbers is even.
13. If n is even then $3n + 6$ is even.

Proofs by Contraposition

1. If n^2 is odd then n is odd.
2. If $\dfrac{\sqrt[3]{x} + 5}{x^2 + 6} = \dfrac{1}{x}$ then $x \neq 8$.
3. If $ab \leq 0$ then $a \leq 0$ or $b \leq 0$.
4. If $a^2 - 2a \neq 0$ then $a \neq 2$.
5. (a) Suppose $n > 0$. If $n = ab$ then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.
   (b) Do not prove, but discuss: why do we only have to check for divisors up to $\sqrt{p}$ to determine whether a number p is prime?
6. $3n + 6$ is even if and only if n is even.
7. $5n + 1$ is even if and only if n is odd.
8. Suppose that $y - x = 3y + x$. If x and y are not both zero, then $y \neq 0$.

Spot the Error

1. I will now prove that if $3x + 4 = 25$ then $x = 7$. Here is the proof: Let $x = 7$. Then $3x = 21$ and so $3x + 4 = 25$. Therefore if $3x + 4 = 25$ then $x = 7$.
2. "Of course Trevor is honest. He told me so himself!"