

Chapters 1.6-1.7: Proofs

Wednesday, June 24

Deductive Reasoning

If Trevor gets stuck in traffic he will be late to work. If he is late to work he will be fired. He will not get stuck in traffic if *and only if* he takes the shortcut.

Which of the following conclusions are logically valid? Prove the ones that are.

1. If Trevor does not take the shortcut then he will be fired.
2. If Trevor takes the shortcut then he will not be fired.
3. If Trevor is not late to work then he took the shortcut.

Use the following notation:

1. S: takes the shortcut
2. T: stuck in traffic
3. L: late to work
4. F: gets fired

Use a visual to help deciding which conclusion is valid—based on the premises given, we can make the following:

$$\neg S \Leftrightarrow T \Rightarrow L \Rightarrow F$$

Taking the contrapositives of the statements gives this:

$$S \Leftrightarrow \neg T \Leftarrow \neg L \Leftarrow \neg F$$

Based on these diagrams, conclusions 1 and 3 are valid but 2 is not. We can prove the first and third in two-column format:

$\neg S$	given
$\neg S \Rightarrow T$	given
T	Modus Ponens
$T \Rightarrow L$	given
L	Modus Ponens
$L \Rightarrow F$	given
F	Modus Ponens

We can prove the third similarly, relying on the contrapositives of the statements:

$\neg L$	given
$T \Rightarrow L$	given
$\neg T$	Modus Tollens
$\neg S \Rightarrow T$	given
S	Modus Tollens

If we lose the game, we will either cry or go out for ice cream (maybe both). Which conclusions are logically valid? Prove the ones that are.

1. If we do not cry and do not go out for ice cream, then we did not lose the game.
2. If we lose the game and do not go out for ice cream, then we will cry.

3. If we do not lose the game and we go out for ice cream, then we will not cry.

Use the following notation:

1. L: Lose the game
2. C: Cry
3. I: Go for ice cream

The original statement can be written as $L \Rightarrow (C \vee I)$. Its contrapositive is $(\neg C \wedge \neg I) \Rightarrow \neg L$, which is precisely conclusion 1.

Conclusion 3 is not valid: we said nothing about what would happen if we won the game. We might cry for joy.

To prove conclusion 2, we will take L and C as premises and then prove I :

Given	Goal
$L \Rightarrow (C \vee I)$ L $\neg I$	C

The proof then goes as follows:

L $L \Rightarrow (C \vee I)$	given
$C \vee I$ $\neg I$	Modus Ponens
C	Disjunctive Syllogism

Variations on a Theorem

Prove:

1. Suppose $a > b$. If $c > 0$ then $ac > bc$. (There is nothing to prove here—this is an axiom!)
2. Suppose $a > b$. If $ac \leq bc$ then $c \leq 0$. This is the contrapositive of (1).
3. Suppose $a > b$. Then $c \leq 0$ or $ac > bc$. Since $(p \vee q)$ is equivalent to $\neg p \Rightarrow q$, this statement is also equivalent to (1).
4. If $c > 0$ and $a \leq b$, then $ac \leq bc$.

...I'm not sure if this can be made totally equivalent to the other statements, but it is the same as "Suppose $c > 0$. If $a \leq b$ then $ac \leq bc$."

We can prove this by cases: If $a = b$ then $ac = bc$. If $a < b$ then $ac < bc$ (by (1)). Therefore if $a \leq b$ then $ac \leq bc$.

5. If $ac \leq bc$ and $c > 0$, then $a \leq b$. The contrapositive of this statement is "If $a > b$ then $c \leq 0$ or $ac > bc$," which is equivalent to (3).

Direct Proofs

Note: Many of these exercises are taken from *How To Prove It*, by Daniel Velleman.

Prove:

1. If $a = 2$ then $a^2 + 2a - 8 = 0$.

Let $a = 2$. Then $a^2 + 2a - 8 = 4 + 4 - 8 = 0$.

2. If $a^2 - 1 = 0$ then $a = 1$ or $a = -1$.

If $a^2 - 1 = 0$ then $(a + 1)(a - 1) = 0$ and so either $a + 1 = 0$ or $a - 1 = 0$. Thus either $a = 1$ or $a = -1$.

3. If $a^2 = b^2$ then $a = b$ or $a = -b$.

If $a^2 = b^2$ then $a^2 - b^2 = 0$ and so $(a + b)(a - b) = 0$. Thus $a + b = 0$ or $a - b = 0$, and so either $a = b$ or $a = -b$ (IOW, $a = \pm b$).

4. If $x^2 = 4$ and $x > 0$ then $x = 2$

By the above result, if $x^2 = 4$ then $x = \pm 2$. Since $x > 0$, $x \neq -2$. Therefore $x = 2$.

5. If $x^2 + y = -3$ and $2x - y = 2$ then $x = -1$.

If $x^2 + y = -3$ and $2x - y = 2$ then (adding the left sides and right sides) we get $x^2 + 2x = -1$.

Therefore, $x^2 + 2x + 1 = (x + 1)^2 = 0$.

Thus $x + 1 = 0$ and so $x = -1$.

6. Suppose $3x + 2y \leq 5$. If $x > 1$, then $y < 1$.

If $x > 1$ then $3x > 3$ and so $3x + 2y > 3 + 2y$.

Since $5 \geq 3x + 2y > 3 + 2y$, we know that $5 > 3 + 2y$.

Therefore $2 > 2y$, and we can conclude that $1 > y$.

7. $3a + 5 = 20$ if and only if $a = 5$.

First prove the forward direction: if $a = 5$ then $3a + 5 = 15 + 5 = 20$, so $3a + 5 = 20$.

Then prove the converse: if $3a + 5 = 20$ then $3a = 15$ and so $a = 5$.

Therefore, $3a + 5 = 20$ if and only if $a = 5$.

8. If $a < b$ then $\frac{a+b}{2} < b$.

When the train of thought behind a series of algebraic steps seems clear, it may be okay to simply write one line above the other. The implication is that we are saying IF one line is true, THEN the following line must also be true.

$$\begin{aligned} a &< b \\ \frac{a}{2} &< \frac{b}{2} \\ \frac{a}{2} + \frac{b}{2} &< \frac{b}{2} + \frac{b}{2} \\ \frac{a+b}{2} &< b \end{aligned}$$

We could write this more explicitly as

- (a) If $a < b$ then $\frac{a}{2} < \frac{b}{2}$.

- (b) If $\frac{a}{2} < \frac{b}{2}$ then $\frac{a}{2} + \frac{b}{2} < \frac{b}{2} + \frac{b}{2}$.
 (c) If $\frac{a}{2} + \frac{b}{2} < \frac{b}{2} + \frac{b}{2}$ then $\frac{a+b}{2} < b$.
 (d) Therefore, if $a < b$ then $\frac{a+b}{2} < b$.

9. Prove that the quadratic formula is correct.

For the sake of saving space, this one will be written out just as a series of lines of algebra:

$$\begin{aligned}
 ax^2 + bx + c &= 0 \\
 x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
 x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} &= \frac{b^2}{4a^2} \\
 \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\
 \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
 x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
 \end{aligned}$$

One important note: going in the forward direction, we have proved that If $ax^2 + bx + c = 0$ THEN $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This is not the same as saying if $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ then $ax^2 + bx + c = 0$. . . we have to separately justify that these two results for x do in fact separately solve the equation $ax^2 + bx + c = 0$.

In this case, we don't have to do any extra work because each of the if-then statements in the proof can be replaced by an if-and-only-if statement. We will go over this more on Thursday.

10. The product of two odd numbers is odd.

Let n and m be odd, so say $n = 2k + 1$ and $m = 2j + 1$. Then $nm = (2k + 1)(2j + 1) = 4jk + 2j + 2k + 1 = 2(2jk + j + k) + 1 = 2o + 1$, and so nm is also odd.

11. The sum of an even number and an odd number is odd. (Note: in this context, "number" means integer)

Let $n = 2k$ be even and $m = 2j + 1$ be odd. Then $nm = 2k(2j + 1) = 2(k(2j + 1)) = 2o$, and so nm is even.

Note that this did not rely on the fact that m was odd. We can do better and say that the product of *any* number and an even number is even:

Let $n = 2k$ and let c be some integer. Then $nc = 2kc = 2(kc)$ and so is even.

12. The product of two even numbers is even.

See previous proof.

13. If n is even then $3n + 6$ is even.

Let $n = 2k$ be even. Then $3n + 6 = 3(2k) + 6 = 6k + 6 = 2(3k + 3)$ and so is even.

Proofs by Contraposition

1. If n^2 is odd then n is odd.

Assuming “even” is equivalent to “not odd,” the contrapositive is “If n is even then n^2 is even,” which we already proved by showing that the product of any number with an even number is even.

2. If $\frac{\sqrt[3]{x} + 5}{x^2 + 6} = \frac{1}{x}$ then $x \neq 8$.

Prove the contrapositive: If $x = 8$ then $\frac{\sqrt[3]{x} + 5}{x^2 + 6} = \frac{2 + 5}{64 + 6} = \frac{7}{70} = \frac{1}{10} \neq \frac{1}{8}$.

Therefore, if $\frac{\sqrt[3]{x} + 5}{x^2 + 6} = \frac{1}{x}$ then $x \neq 8$.

3. If $ab \leq 0$ then $a \leq 0$ or $b \leq 0$.

The contrapositive of this statement is “if $a > 0$ and $b > 0$ then $ab > 0$,” which is axiomatically true.

4. If $a^2 - 2a \neq 0$ then $a \neq 2$.

Prove the contrapositive: If $a = 2$ then $a^2 - 2a = 4 - 4 = 0$.

5. (a) Suppose $n > 0$. If $n = ab$ then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Contrapositive: If $a > \sqrt{n}$ and $b > \sqrt{n}$ then $ab > \sqrt{n}\sqrt{n} = n$.

- (b) Do not prove, but discuss: why do we only have to check for divisors up to \sqrt{p} to determine whether a number p is prime?

If p has a divisor greater than \sqrt{p} , then by the above result it also has one less than \sqrt{p} . So if we check up to \sqrt{p} we would find the smaller one. If we find nothing, then there are no divisors greater than \sqrt{p} either.

6. $3n + 6$ is even if and only if n is even.

Say the original statement is “If $3n + 6$ is even, then n is even.” We get the following variations on the statement:

Contrapositive: If n is odd then $3n + 6$ is odd.

Inverse: If $3n + 6$ is odd then n is odd.

Converse: If n is even then $3n + 6$ is even.

In order to prove the biconditional we must prove (1) either the original statement or its contrapositive, and (2) either the inverse or the converse. We will prove the converse and the contrapositive.

First, we can prove the directly that if n is even then $3n + 6$ is even (the converse): if $n = 2k$ then $3n + 6 = 6k + 6 = 2(3k + 3)$.

Next, we prove the contrapositive: If $n = 2k + 1$ is odd then $3n + 6 = 6k + 9 = 2(3k + 4) + 1$ is also odd.

Since we have proved the converse and the contrapositive, we know that $3n + 6$ is even if and only if n is even.

7. $5n + 1$ is even if and only if n is odd.

Let the base statement be “If $5n + 1$ is even then n is odd.” We will prove the converse and contrapositive.

Prove the converse: If $n = 2k + 1$ is odd then $5n + 1 = 5(2k + 1) + 1 = 10k + 6 = 2(5k + 3)$ is even.

Contrapositive: If $n = 2k$ is even then $5n + 1 = 10k + 1$ is odd.

8. Suppose that $y - x = 3y + x$. If x and y are not both zero, then $y \neq 0$.

As it turns out, the first premise is unnecessary. It is a standard rule of inference that if $x \neq 0$ and $y \neq 0$ then $y \neq 0$.

But if we *were* to prove it using the contrapositive, we would prove that if $y = 0$ then x and y are both zero, as follows:

If $y = 0$ then $0 - x = 3 \cdot 0 + x$, and so $-x = x$. Therefore $2x = 0$, implying that $x = 0$. So $x = 0$ and $y = 0$.

Spot the Error

1. I will now prove that if $3x + 4 = 25$ then $x = 7$. Here is the proof: Let $x = 7$. Then $3x = 21$ and so $3x + 4 = 25$. Therefore if $3x + 4 = 25$ then $x = 7$.

I proved that if $x = 7$ then $3x + 4$ is 25—the converse of the original statement, which is not equivalent. As it turns out, I could say “If and only if” at each step:

$$3x + 4 = 25 \Leftrightarrow 3x = 21 \Leftrightarrow x = 7,$$

which proves both directions of the statement in one go.

2. “Of course Trevor is honest. He told me so himself!”

This is an example of *begging the question*: the reasoning that is used to conclude that Trevor is honest depends on the assumption that Trevor is honest:

- (a) Trevor told me that he is honest.
- (b) What Trevor says is true *because Trevor is honest*.
- (c) So Trevor’s statement that he is honest is true.
- (d) Therefore, Trevor is honest.