Chapters 1.3-1.6: Rules of Inference

Tuesday, June 23

1. Show that the following equivalences with truth tables:

(a) $\mathbf{T} \vee p \equiv \mathbf{T}$

p	$\mathbf{T}$	$\mathbf{T} \vee p$
T	T	T
F	T	T

(b) $\mathbf{F} \wedge p \equiv \mathbf{F}$

p	$\mathbf{F}$	$\mathbf{F} \wedge p$
T	F	F
F	F	F

(c) $(p \vee \neg p) \equiv \mathbf{T}$

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(d) $\mathbf{F} \vee p \equiv p$

$\mathbf{F}$	p	$\mathbf{F} \vee p$
F	T	T
F	F	F

(e) $p \Rightarrow \mathbf{T} \equiv \mathbf{T}$

p	$\mathbf{T}$	$p \Rightarrow \mathbf{T}$
T	T	T
F	T	T

(f) $p \Rightarrow \mathbf{F} \equiv (\neg p)$

p	$\mathbf{F}$	$p \Rightarrow \mathbf{F}$	$\neg p$
T	F	F	F
F	F	T	T

2. A traveler comes upon a snake, who says this: "If you move, I will strike. If you do not move, I will strike." Write this as a single compound proposition and show that it is logically equivalent to "I will strike." Which rule of inference applies here?

M: "You move"

S: "I will strike"

As a proposition, this is either $(M \Rightarrow S) \wedge (\neg M \Rightarrow S)$ or $(M \vee \neg M) \Rightarrow S$ (Show that the two are equivalent!) For a rule of inference, we can use resolution, and the fact that $(S \vee S) \equiv S$:

$$\begin{array}{rl} M \Rightarrow S & \\ \underline{\neg M \Rightarrow S} & \\ \underline{S \vee S} & \text{(Resolution)} \\ S & \end{array}$$

Or going from the second variation we could say (since $M \vee \neg M$ is always true):

$$\begin{aligned} (M \vee \neg M) \Rightarrow S &\equiv \mathbf{T} \Rightarrow S \\ &\equiv S \end{aligned}$$

3. Show that $((p \Rightarrow q)$ and $(q \Rightarrow r)) \Rightarrow$ (p$\Rightarrow r$) is a tautology by at least two different methods.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Trying to do this just through logical equivalences would be very unpleasant, so break this down by whether p is true or false. If p is true we get the following:

$$\begin{aligned} ((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r) &\equiv ((\mathbf{T} \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (\mathbf{T} \Rightarrow r) \\ &\equiv (q \wedge (q \Rightarrow r)) \Rightarrow r, \end{aligned}$$

which is the tautology for Modus Ponens (if p and (p $\Rightarrow$ q) then q) and therefore always true. If p is false, we get

$$\begin{aligned}((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r) &\equiv ((\mathbf{F} \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (\mathbf{F} \Rightarrow r) \\ &\equiv (\mathbf{T} \wedge (q \Rightarrow r)) \Rightarrow \mathbf{T} \\ &\equiv (q \Rightarrow r) \Rightarrow \mathbf{T} \\ &\equiv \mathbf{T}\end{aligned}$$

4. Show that $p \Rightarrow (q \Rightarrow r), q \Rightarrow (p \Rightarrow r)$, and $(p \wedge q) \Rightarrow r$ are all equivalent.

   We could do with with truth tables, but will instead use a series of equivalences–in particular, (1) $(p \Rightarrow q) \equiv (\neg p \vee q)$ and (2) one of De Morgan's laws.

$$\begin{aligned}p \Rightarrow (q \Rightarrow r) &\equiv p \Rightarrow (\neg q \vee r) && (1) \\ &\equiv \neg p \vee (\neg q \vee r) && (1) \\ &\equiv (\neg p \vee \neg q) \vee r \\ &\equiv \neg(p \wedge q) \vee r && (2) \\ &\equiv (p \wedge q) \Rightarrow r && (1)\end{aligned}$$

   This shows that the first and third propositions are equivalent. Then since both of them are equivalent to $\neg p \vee \neg q \vee r$ (which was shown in the third line), we can show the third equivalence as follows:

$$\begin{aligned}\neg p \vee \neg q \vee r &\equiv \neg q \vee (\neg p \vee r) \\ &\equiv q \Rightarrow (\neg p \vee r) && (1) \\ &\equiv q \Rightarrow (p \Rightarrow r). && (1)\end{aligned}$$

   Therefore, all three propositions are equivalent.

Back on the island of Knights and Knaves: recall that knights are virtuous and always tell the truth while knaves are wicked and always lie. There are also werewolves, who may be either knights or knaves.

You meet three people, A, B, and C. You are looking for a traveling partner and would rather go with a knight than a knave, but above all you want to avoid going with a werewolf (for obvious reasons). You happen to know that exactly one of the three islanders is a werewolf, but don't know which. Based on their statements, who should you pick as a companion?

5. A: Exactly one of us is a knave.
   B: Exactly two of us are knaves.
   C: All three of us are knaves.

   The three statements are all contradictory, so at least two of them are knaves. This means that either B or C is telling the truth, so there is at least one knight. Thus there are exactly one knight and two knaves, and so B is the knight. However, you have no way of knowing which one is the werewolf.

6. A: At least one of us is a knight.
   B: At least one of us is a knave.
   C: The werewolf is a knight.

   If B were a knave then his statement would be true, which is a contradiction. Thus B is a knight. A is telling the truth and is therefore also a knight. Since B (a knight) said that there was at least one knave, C must be the knave. Since C is lying, the werewolf is a knave. Since A and B are knights, C is the werewolf.

7. A: I am not the werewolf.
   B: The werewolf is a knave.
   C: All three of us are knaves.
   A: C is a knave!

   If C were a knight then his statement would imply that he is a knave, a contradiction. Thus C is knave and A (saying so) is a knight. A is not the werewolf, by his own testimony.

   If B were a knave, then the werewolf would be a knight, but we already know that A (the only knight) is not a werewolf. So B must be a knight, and so C is (once again) the werewolf.

8. Two students are wearing hats. Each can see the other's hat but not their own. The instructor informs them that each of them is wearing either a red hat or a white hat, and that at least one of them is wearing a red hat. The instructor then asks if either of the students knows the color of their own hat.

   Student 1: No, I don't know.
   Student 2: Well, I didn't know at first, but now I do!

   What are the colors of the students' hats? How did Student 2 know?

   If Student 2 were wearing a white hat, then Student 1 would have immediately known that his own hat was the red one. Since Student 2 did not know, Student 2 deduces that she is wearing a red hat.

   By the same logic, if Student 1 were wearing a white hat then Student 2 would have known immediately that her hat was red. But she says "I didn't know at first," implying that Student 1 is also wearing a red hat. (That is, this works if we assume that she meant that she didn't know until Student 1 gave his answer.)

9. Another hat problem: this time there are three students with hats. They are sitting in a straight line so that the student in back can see the other two, the middle student can see the front student, and the front student can't see anything. They are told that they are each wearing a white or a red hat, and that at least one of them is wearing a red hat.

The back student says “I do not know the color of my hat.”

Next, the middle student says “I do not know the color of my hat.”

The front student then says “I know the color of my hat!”

What color hats are the students wearing, and how does the front student know?

If the front two students were both wearing white hats then the back student would know that he was wearing a red hat. Thus at least one of the two students in front is wearing a red hat.

The middle student, hearing the student in back, deduces as much. So if the front student were wearing a white hat the middle student would know that her own hat was red. Since she does not know, the front student must be wearing a red hat. Following this same deduction process, the front student can know the color of his own hat.

We cannot tell the colors of the hats of the other two students.