Math 55: Practice Midterm 3
Midterm: Friday, July 31

1. 50 people go out to eat. Everyone orders either a hamburger or a salad. 15 people put mustard on their burgers, 25 put ketchup on their burgers, and 10 people put both ketchup and mustard on their burgers. How many people ordered a salad?

   Oops, not enough information. We need to assume that this "or" is exclusive and that everone who got a burger put ketchup or mustard on it. Then the number who got burgers is $15 + 25 - 10 = 30$, and so 50 - 30 = 20 people got the salad.

2. How many times must I roll a pair of dice in order to guarantee that I roll some number (the sum of the two dice) twice?

   There are 11 possible totals for rolling a pair of dice and so 12 rolls will guarantee that some number is rolled twice (or more).

3. How many ways are there to put 3 red chairs and 4 blue chairs around a circular table if chairs of the same color are indistinguishable and two arrangements that differ only by rotating the table count as the same?

   There are $\binom{7}{3} = 35$ ways to arrange the chairs if the orientation of the table matters, and so there are $35/7 = 5$ ways to do it where the orientation of the table does not matter.

   Note that this depends on the fact that no particular arrangement is formed twice just in rotating the table 7 times. If we were instead arranging 2 red and 2 blue chairs around a table then there would only be 2 ways to do so (red chairs opposite or adjacent), but using naive division would give $\binom{4}{2}/4 = 1.5$ ways, which is impossible. This is wrong because if the red chairs are opposite each other, rotating the table twice gives the exact same pattern. Thus using division does not accurately count the number of symmetries.

4. How many distinct ways are there to put 2 red dots and 4 blue dots on the faces of a (blank and symmetrical) cube so that each face gets one dot?

   Enumeration of the possibilities is the best way to go here... using the division rule leads to complications for reasons similar to those described above. The red dots are adjacent to each other or they aren't, so there are only 2 possibilities.

5. Evaluate the sum $\sum_{i=0}^{20} \binom{n}{i}(-1)^i 2^{n-i}$

   By the Binomial Theorem this is equal to $(-1+2)^{20} = 1^{20} = 1$.

6. I have 3 teal balls, 4 magenta balls, and 5 orange balls in a cauldron. If I draw 3 balls without replacement, what is the probability that I get 2 orange balls and 1 magenta? What if I draw 3 balls with replacement?

   Without replacement the number of ways to draw such balls is $\binom{5}{2}\binom{4}{1} = 40$ and the total number of possibilities is $\binom{12}{3} = 220$, so the probability is $2/11 \approx .182$.

   With replacement there are 3 possibilities: OOM, OMO, and MOO. The probability of each is $(5/12)^2 \cdot (4/12)$, so the total probability is $3 \cdot (5/12)^2 \cdot 4/12 = 25/144 \approx .174$. This particular color combination is marginally more likely if you draw without replacement.

7. How many ways are there to give 8 cookies to 4 friends if every friend must get at least 1 cookie?

   Give 1 cookie to each friend, there are now 4 "extra" cookies to give to 4 friends where each friend requires a cookie. Using the Stars and Bars method, the number of ways to give out the cookies is $\binom{7}{3} = 35$.

8. How many ways are there to buy 7 fruit if my options are apples, bananas, and peaches?

   Stars and bars again: the number of ways is $\binom{7+3-1}{3-1} = \binom{9}{2} = 36$.

9. How many ways are there to give 5 blue hats, 2 red hats, and 3 green hats to 10 friends?

   The number of ways is $\frac{10!}{2!3!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{12} = 2520$.

   Also equal to $\binom{10}{2}\binom{8}{3} = 45 \cdot 56 = 2520$.

10. If I am dealt a random hand of 5 cards, what is the probability of getting a straight (e.g. 2-3-4-5-6 or 8-9-10-J-Q in any combination of suits; A-2-3-4-5 is not okay but 10-J-Q-K-A is fine)?

    There are 9 ways to choose the smallest card in the straight and then 4^5 ways to choose the particular cards that make up that straight, so the total number of ways is $\frac{9 \cdot 4^5}{\binom{52}{5}} \approx .003546$. This is a little larger than the probability of getting a "straight" in poker since straight flushes and royal flushes are counted as their own categories.

11. There is a 50% chance that it rains tomorrow and a 30% chance that I will go outside. If these are independent events, what is the chance that it rains but I stay inside?

    $p(R \cap \overline{O}) = p(R)p(\overline{O}) = p(R)(1 - p(O)) = .5 \cdot .7 = .35$.

12. A fair coin and a loaded coin (p(heads) = .7) are sitting on a table. If I take a random coin and flip 6 heads out of 10, what is the chance that I took the fair coin?

    Let F be the chance of a fair coin and S be the sequence of flips. Using Bayes' Theorem, the probability that the coin is fair is

    $$p(F|S) = \frac{p(S|F)p(F)}{p(S|F)p(F) + p(S|\overline{F})p(\overline{F})}$$
    $$= \frac{\binom{10}{6}.5^{10}(1/2)}{\binom{10}{6}.7^6.3^4(1/2) + \binom{10}{6}.5^{10}(1/2)}$$
    $$= \frac{.5^{10}}{.7^6.3^4 + .5^{10}}$$
    $$= \frac{1}{1 + (7/5)^6(3/5)^4}$$
    $$\approx .506$$

    It is barely more likely that the coin was fair given this outcome.

13. I roll two dice. If X is the sum of the rolls and Y is the product of the rolls, prove that X and Y are not independent random variables.

    $p(X = 2 \cap Y = 1) = 1/36$ but $p(X = 2)p(Y = 1) = 1/36^2$. Since these two values are not equal, X and Y are not independent.

14. I flip a coin five times and get 2^n dollars for every time I flip heads. What is the expected amount of money I will make in the game?

The amount of money I will expect to make is $\frac{1}{32}(1\cdot1+2\cdot5+4\cdot10+8\cdot10+16\cdot5+32\cdot1) = 243/32 \approx 7.59$.

You can also find the above sum using the Binomial Theorem: $\sum_{i=0}^{5} 2^i 1^{5-i} = (2+1)^5 = 3^5 = 243$.

15. I roll a die, multiply the result by 3, then add 7. What is the expected value of my final number? What is the variance?

The expected value of a die is 3.5, so $E(3D+7) = 3E(D)+7 = 3\cdot3.5+7 = 17.5$.

The variance of the roll of a die is $35/12$, so $Var(3D+7) = 9Var(D) = 105/4$.

16. Prove that if E and F are independent events then $\overline{E}$ and F are also independent events.

First note that $p(E\cap F)+p(\overline{E}\cap F) = p((E\cup\overline{E})\cap F) = p(F)$. Therefore $p(\overline{E}\cap F) = p(F)-p(E\cap F) = p(F)-p(E)p(F) = p(F)(1-p(E)) = p(F)p(\overline{E})$.

17. Prove that if E and F are independent random variables and $G = 2E+3$ then G and F are independent random variables.

For any $r, s \in \mathbb{R}$, we have

$$
\begin{aligned}
p(G = r \cap F = s) &= p(2E+3 = r \cap F = s) \\
&= p(E = (r-3)/2 \cap F = s) \\
&= p(E = (r-3/2))p(F = s) \\
&= p(G = r)p(F = s)
\end{aligned}
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