

Math 55: Practice Midterm 2

Midterm: Friday, July 17

1. Find $\sum_{i=1}^2 \prod_{j=1}^3 (i+j)$

$$\begin{aligned}\sum_{i=1}^2 \prod_{j=1}^3 (i+j) &= \prod_{j=1}^3 (1+j) + \prod_{j=1}^3 (2+j) \\ &= (2 \cdot 3 \cdot 4) + (3 \cdot 4 \cdot 5) \\ &= 24 + 60 \\ &= 84\end{aligned}$$

2. Define $a_1 = 1$ and for $n \geq 2$ define $a_n = \sum_{i=1}^{n-1} a_i$.

(a) Find a_5 .

$$a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 4, a_5 = 8$$

(b) Find a formula for a_n where $n \geq 2$ and prove that it is correct.

Conjecture that $a_n = 2^{n-2}$ for $n \geq 2$. Proof by strong induction:

Base case: when $n = 2, a_2 = 1 = 2^0$.

Inductive step: Suppose that $a_k = 2^{k-2}$ for $2 \leq k \leq n$. Then

$$\begin{aligned}a_{n+1} &= \sum_{i=1}^n a_i \\ &= a_1 + \sum_{i=2}^n 2^{i-2} \\ &= 1 + \sum_{j=0}^{n-2} 2^j \\ &= 1 + (2^{n-1} - 1) \\ &= 2^{n-1}\end{aligned}$$

3. Find the greatest common divisor of 184 and 306.

$$\begin{aligned}\gcd(184, 306) &= \gcd(184, 306 - 184) \\ &= \gcd(184, 122) \\ &= \gcd(62, 122) \\ &= \gcd(62, 60) \\ &= \gcd(2, 60) \\ &= 2\end{aligned}$$

4. Find a solution to $184x + 306y = d$, where $d = \gcd(184, 306)$.

Rewrite the Euclidean algorithm and reverse as follows (taking some shortcuts to reduce the number of steps):

$$\begin{aligned}306 - 2 * 184 &= -62 \\ 184 - 3 * 62 &= -2 \\ 184 + 3 * (306 - 2 * 184) &= -2 \\ 5 * 184 - 3 * 306 &= 2\end{aligned}$$

5. Solve: $184x \equiv 16 \pmod{306}$.

The previous solution gave $5 \cdot 184 \equiv 2 \pmod{306}$, so multiplying both sides by 8 gives $40 \cdot 184 \equiv 16 \pmod{306}$. Solution: $x \equiv 40 \pmod{306}$.

6. Find all solutions to $36x \equiv 17 \pmod{60}$.

36 and 60 are both divisible by 6 but 17 is not. No solutions.

7. Find all solutions to $36x \equiv 18 \pmod{60}$.

Divide all numbers by 6 to get $6x \equiv 3 \pmod{10}$. 6 and 10 are both even but 3 is odd. Still no solutions.

8. Evaluate the following:

(a) $815 \pmod{7}$

$815 = 116 \cdot 7 + 3$, so $815 \equiv 3 \pmod{7}$.

(b) $23234 \cdot 101 \pmod{4}$

$23234 \cdot 101 \equiv 2 \cdot 1 \equiv 2 \pmod{4}$.

(c) $(-17) \cdot 82 \pmod{3}$

$(-17) \cdot 82 \equiv 1 \cdot 1 \equiv 1 \pmod{3}$

(d) $5^{88} \pmod{6}$

$5^{88} \equiv (-1)^{88} \equiv 1 \pmod{6}$.

(e) $98 \cdot 96 \pmod{99}$

$98 \cdot 96 \equiv (-1) \cdot (-3) \equiv 3 \pmod{99}$.

(f) $2^{87} \pmod{7}$

By FLT, $2^6 \equiv 1 \pmod{7}$. So $2^{87} \equiv 2^{6 \cdot 14 + 3} \equiv (2^6)^{14} \cdot 2^3 \equiv 1 \cdot 1 \equiv 1 \pmod{7}$.

(g) $2^{87} \pmod{35}$

Solve mod 5 and mod 7 separately, using FLT. $2^{87} \equiv 2^3 \equiv 3 \pmod{5}$ and $2^{87} \equiv 1 \pmod{7}$, so solving the two congruences simultaneously gives $2^{87} \equiv 8 \pmod{35}$.

9. Prove that 101 is prime.

101 is not divisible by 2,3,5,7, or 11. We don't have to check any further since if a number n is composite it has a divisor less than or equal to \sqrt{n} .

10. Say whether each of the following equations has an integer solution:

(a) $x \equiv 18 \pmod{45}, x \equiv 1 \pmod{2}$

Yes by CRT, since $\gcd(45, 2) = 1$.

(b) $x \equiv 13 \pmod{15}, x \equiv 5 \pmod{6}$

No, since the first equation implies $x \equiv 1 \pmod{3}$ but the second implies $x \equiv 2 \pmod{3}$.

(c) $x \equiv 12 \pmod{15}, x \equiv 3 \pmod{6}$

Yes, since we can turn this into the three equations $x \equiv 2 \pmod{5}$, $x \equiv 0 \pmod{3}$, and $x \equiv 1 \pmod{2}$. Then by the CRT, there exists a unique solution mod 30.

11. Prove that if $a \equiv b \pmod{m}$ then $-a \equiv m - b \pmod{m}$.

If $a \equiv b \pmod{m}$ then $m \mid (a - b)$, so $m \mid (b - a) - m = (-a) - (m - b)$. Therefore $-a \equiv m - b \pmod{m}$.

Or: If $a \equiv b \pmod{m}$ then there is some k such that $a = b + km$. Then $-a = -b - km = m - b - (k+1)m$, so there exists some $j \in \mathbb{Z}$ such that $-a = (m - b) + jm$. Therefore $-a \equiv m - b \pmod{m}$.

12. Prove that the equation $x^2 + 3x + 5y = 1$ has no solutions where x and y are integers.

This is equivalent to the statement that $x^2 + 3x \equiv 1 \pmod{5}$ has no solutions for any integer x . We just need to check the five cases $x \equiv 0, 1, 2, 3, 4 \pmod{5}$, which give the answers 0, 4, 0, 3, 3 $\pmod{5}$. None of these are 1, so the equation has no integer solutions.

13. How many divisors does 100 have?

$100 = 2^2 5^2$, so 100 has $(2+1)(2+1) = 9$ divisors. (These are 1, 2, 4, 5, 10, 20, 25, 50, and 100.)

14. How many numbers are relatively prime to 100?

50 numbers are divisible by 2 and only numbers ending in 5 are divisible by 5 but not 2. There are 10 of those, so $100 - 50 - 10 = 40$ numbers are relatively prime to 100.

Using a formula for $\varphi(100)$, we can also say $\varphi(100) = 100 \cdot (1 - 1/2)(1 - 1/5) = 40$.

15. How many zeroes does $100!$ end with?

This would be equal to the number of fives in the prime factorization of $100!$, which is equal to $\lfloor 100/5 \rfloor + \lfloor 100/25 \rfloor = 20 + 4 = 24$.