Math 55: Practice Midterm 1

Midterm: Thursday, July 2

1. Prove using truth tables that $\neg(p \wedge q) \wedge p$ is equivalent to $\neg q \wedge p$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg(p \wedge q) \wedge p$	$\neg q \wedge p$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	T	F	F

2. Let w be the statement "It is Wednesday," d be "I have a dollar," and s be "I can buy a shake." Write the following using w, d, s, and logical connectives:

   (a) I can buy a shake if today is Wednesday and I have a dollar.

   $$(w \wedge d) \Rightarrow s$$

   (b) Today is Wednesday, but I do not have a dollar.

   $$w \wedge \neg d$$

   (c) I need a dollar in order to buy a shake.

   $$\neg d \Rightarrow \neg s, \text{ or } s \Rightarrow d$$

3. Assume the previous three statements are all true. Prove the following:

   This was written a little sloppily–you should be able to do these questions using only the first and third statements.

   (a) If I cannot buy a shake but I have a dollar, then today is not Wednesday.

   In order to prove this, we will take "I cannot buy a shake" and "I have a dollar" as given and prove "Today is not Wednesday." In addition, use the fact that the contrapositive of 1 is $\neg s \Rightarrow (\neg w \vee \neg d)$.

$\neg s$	given
$\neg s \Rightarrow (\neg w \vee \neg d)$	contrapositive of (1)
$\neg w \vee \neg d$	Modus Ponens
d	given
$\neg w$	Disjunctive Syllogism

   (b) If today is Wednesday, then I can buy a shake if and only if I have a dollar.

   We have to prove two statements: "If today is Wednesday and I have a dollar, I can buy a shake", and "If today is Wednesday and I do not have a dollar, I cannot buy a shake."

   The first is precisely what (1) says. The second is true because of (3).

4. Draw a Venn Diagram showing the relation between $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and S the set of all numbers divisible by 3.

   Your diagram should show $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. In addition, $S \subset \mathbb{Z}$, and the intersection between S and $\mathbb{N}$ is non-zero. However, there is space inside $\mathbb{Z}$ that is not covered by $S \cup \mathbb{N}$.

5. What is the contrapositive of "If today is Tuesday or Wednesday then pizza is on sale"? What about the inverse and converse?

(a) Contrapositive: "If pizza is not on sale then today is not Tuesday and not Wednesday."
(b) Inverse: "If today is not Tuesday and not Wednesday then pizza is not on sale."
(c) Converse: "If pizza is on sale, then today is Tuesday or Wednesday."

Remember that the inverse and converse are equivalent to each other, but only the contrapositive is equivalent to the forward statement.

6. Let $A = \{1, 2, 3\}$, $B = \{2, 4\}$, and let the universe U be $\{1, 2, 3, 4\}$. Describe the following using only A and B and set operations:

   (a) $\{1, 3\} = A - B = \overline{B}$
   (b) $\{2\} = A \cap B$
   (c) $\{1, 3, 4\} = \overline{A \cap B} = \overline{A} \cup \overline{B}$
   (d) $\{4\} = B - A$

7. Draw a Venn Diagram, and prove: if $A \subset B$ then $\overline{B} \subset \overline{A}$.

   No Venn Diagram, sorry.

   Given: $A \subset B$.

   Prove: $\overline{B} \subset \overline{A}$.

   Proof: The statement "$A \subset B$" is equivalent to the statement "If $x \in A$ then $x \in B$," which is equivalent to "If x $\notin B$ then $x \notin A$" by the contrapositive, which is equivalent by definition to $\overline{B} \subset \overline{A}$.

8. Prove that $3a + 2$ is even if and only if a is even for integers a.

   If $a = 2k$ is even then $3a + 2 = 6k + 2 = 2(3k + 1)$, which is even.

   If $a = 2k + 1$ is odd then $3a + 2 = 6k + 3 + 2 = 6k + 5 = 2(3k + 2) + 1$, which is odd.

   Therefore, $3a + 2$ is even if and only if a is even.

9. Prove that for any integer n, $n(n^2 - 1)(n^2 + 1)$ is divisible by 5. (You do not need to expand the entire expression! Just show that at least one of the three factors is divisible by 5).

   (a) If $n = 5k$, then we are done.
   (b) If $n = 5k + 1$ then $n^2 - 1 = (n + 1)(n - 1) = (n + 1)(5k)$, which is divisible by 5.
   (c) If $n = 5k + 2$ then $n^2 + 1 = (5k + 2)^2 + 1 = 25k^2 + 20k + 5$, which is divisible by 5.
   (d) If $n = 5k + 3$ then $n^2 + 1 = (5k + 3)^2 + 1 = 25k^2 + 30k + 10$, which is divisible by 5.
   (e) If $n = 5k + 4$ then $n^2 - 1 = (5k + 4)^2 - 1 = 25k^2 + 40k + 15$, which is divisible by 5.

10. Express in quantifier notation, and prove or disprove:

    (a) There is no smallest positive real number a.

    $$\neg(\exists a > 0)(\forall x > 0)(a \leq x)$$

    $$(\forall a > 0)(\exists x > 0)(x < a)$$

    (b) The interval $[0, 2]$ has a largest element.

    $$(\exists x \in [0, 2])(\forall y \in [0, 2])(y \leq x)$$

    In this case, 2 is the desired largest element.

    (c) The square of any real number is positive.

    $$(\forall r \in \mathbb{R})(r^2 > 0)$$

    This statement is false, with r=0 serving as a counterexample.