NAME: ____________________________

1. Show that the expression \((p \Rightarrow q) \Rightarrow (q \Rightarrow p)\) is neither a tautology nor a contradiction.

   If \(p\) is false but \(q\) is true then the entire expression is \((F \Rightarrow T) \Rightarrow (T \Rightarrow F) \equiv T \Rightarrow F \equiv F\). Otherwise the entire expression is true.

2. State the negation and prove or disprove: \((\forall x)(\exists y)(\forall z)(xy \geq z)\)

   The negation: \((\exists x)(\forall y)(\exists z)(xy < z)\). The original statement is false, and we will show it by proving
   the negation: let \(x = 0\). Then for any \(y\), let \(z = 1\). Then \(xy = 0 < 1 = z\).

3. Prove that if \(x\) and \(y\) are positive then \(\sqrt{\frac{x^2 + y^2}{2}} \geq \frac{x + y}{2}\).

   Work backwards from the conclusion:
   \[
   \frac{\sqrt{x^2 + y^2}}{2} \geq \frac{x + y}{2} \iff \frac{x^2 + y^2}{2} \geq \frac{x^2 + 2xy + y^2}{4} \\
   \iff 2x^2 + 2y^2 \geq x^2 + 2xy + y^2 \\
   \iff x^2 - 2xy + y^2 \geq 0 \\
   \iff (x - y)^2 \geq 0
   \]

   The last inequality is true for all \(x\) and \(y\) and the inequalities are connected as if-and-only-if statements, so
   the inequality that we desired to prove is also true.

4. Evaluate: \(\sum_{i=1}^{10} \sum_{j=1}^{i} i - 2j\)

   \[
   \sum_{i=1}^{10} \sum_{j=1}^{i} i - 2j = \sum_{i=1}^{10} \left( i^2 - 2 \sum_{j=1}^{i} j \right) \\
   = \sum_{i=1}^{10} \left( i^2 - 2 \cdot \frac{i^2 + i}{2} \right) \\
   = \sum_{i=1}^{10} -i \\
   = -55
   \]

5. Find integers \(x, y \in \mathbb{Z}\) such that \(18x + 40y = 14\)

   \(a\) Method 1: Find the gcd and solve \(18x + 40y = 2\), then multiply by 7:
   \[
   40 - 2 \cdot 18 = 4 \\
   18 - 4 \cdot 4 = 2 \\
   18 - 4 \cdot (40 - 2 \cdot 18) = 2 \\
   9 \cdot 18 - 4 \cdot 40 = 2 \\
   63 \cdot 18 - 28 \cdot 40 = 14
   \]
(b) Method 2: Trial and error. \[ 40 - 2 \cdot 18 = 4 \text{ and } 18 - 4 = 14, \text{ so } 18 - (40 - 2 \cdot 18) = 3 \cdot 18 - 40 = 14. \]

Note also that \[ 20 \cdot 18 - 9 \cdot 40 = 0, \text{ so the two solutions (63,-28) and (3, -1) differ by a multiple of (20,-9).} \]

6. Determine whether each of the systems of equations has a solution:

(a)

\[
\begin{align*}
x &\equiv 15 \pmod{35} \\
x &\equiv 8 \pmod{10} \\
x &\equiv 1 \pmod{7}
\end{align*}
\]

The first congruence implies that \( x \equiv 0 \pmod{5} \) and the second that \( x \equiv 3 \pmod{5} \), so there is no solution.

(b)

\[
\begin{align*}
x &\equiv 3 \pmod{6} \\
x &\equiv 7 \pmod{8} \\
x &\equiv 4 \pmod{5}
\end{align*}
\]

The first two congruences are the only ones involving non-relatively-prime numbers. The first implies that \( x \equiv 1 \pmod{2} \) and the second that \( x \equiv 1 \pmod{2} \), and so are compatible. By the Chinese Remainder Theorem, the solution will be unique modulo \( 3 \cdot 8 \cdot 5 = 120 \).

7. Prove using induction that if \( G \) is a tree with at least 2 vertices then \( \chi(G) = 2 \). You may use the fact that every tree with 2 or more vertices has at least 2 vertices of degree 1.

Base case: If \( G \) has 2 vertices then \( \chi(G) = 2 \).

Inductive step: suppose that any tree with \( n \) vertices can be colored, and consider a tree with \( n + 1 \) vertices. At least one of these vertices has degree 1, so remove it for now. What remains is a tree with \( n \) vertices, which by the inductive hypothesis can be colored with 2 colors. Now replace the vertex that was removed, and color it the color opposite its lone neighbor.

8. State the inverse, converse, and contrapositive, and prove or disprove each one: “If a number is divisible by 4 and 5 then it is divisible by 20.”

Forward statement: true. If \( 4|n \) and \( 5|n \), then the fact that \( \gcd(4, 5) = 1 \) implies that \( 4 \cdot 5|n \).

Inverse: If a number is not divisible by 4 or not divisible by 5 then it is not divisible by 20. True because the inverse is equivalent to the converse.

Converse: If \( 20|n \) then \( 4|n \) and \( 5|n \). True because if \( 20|n \) then \( 20k = n \) for some \( k \), but \( 20k = 5(4k) = 4(5k) \).

9. I draw cards from a deck until I have drawn all 4 aces. What is the expected number of kings that I will have drawn?

Let \( E_i \) be the event that the \( i \)-th king comes before the last ace. The chance of this is \( 4/5 \), since there is a \( 1/5 \) chance that a random shuffle of the cards AAAAK will end in the king.

There are 4 kings, so the total expected number of kings drawn will be \( 4 \cdot (4/5) = 16/5 \).

10. Prove that if the events \( E \) and \( F \) are positively correlated then the events \( E \) and \( \overline{F} \) are negatively correlated.

We will do a proof by contradiction: say that \( p(E \cap F) > p(E)p(F) \) but \( p(E \cap \overline{F}) \geq p(E)p(\overline{F}) \). Then \( p(E) = p(E \cap (F \cup \overline{F})) = p((E \cap F) \cup (E \cap \overline{F})) = p(E \cap F) + p(E \cap \overline{F}) > p(E)p(F) + p(E)p(\overline{F}) = p(\overline{F}) \).
This is a contradiction, so our assumption that \( p(E \cap F) \geq p(E)p(F) \) must have been incorrect. Therefore if \( E \) and \( F \) are positively correlated then \( E \) and \( F \) are negatively correlated.

11. 10 cows, 10 ducks, and 10 pigs are all standing in a line, their positions distributed at random. What is the expected number of times a cow will be standing directly in front of a duck?

Let \( E_{i,j} \) be the event that cow \( i \) is directly in front of duck \( j \). The probability of each such event is \( \frac{29}{30 \cdot 29} = \frac{1}{30} \). There are 100 such events so the total expected value is \( \frac{100}{30} = \frac{10}{3} \).

12. If I flip a fair coin 40 times, prove that the probability of getting 30 or more heads is less than or equal to 1/20.

The variance is \( np(1-p) = \frac{40}{4} = 10 \), so by Chebyshev’s inequality \( p(|H - 25| \geq 10) = \frac{10}{10^2} = \frac{1}{10} \). This is the probability of getting at least 30 or at most 20 heads, so the probability of getting just 30 or more heads is half that: \( \frac{1}{20} \).

13. There is an urn with 5 red balls and 3 yellow balls. I draw 2 balls from the urn, flipping a fair coin to decide whether to draw with or without replacement. If I draw 1 red ball and 1 yellow, what is the probability that I drew without replacement?

If I draw without replacement then the chance of getting 1 red and 1 yellow is \( \frac{15}{8} \cdot \frac{15}{32} = \frac{15}{12} = \frac{15}{28} \). If I draw with replacement the chance is \( 2 \cdot \frac{3}{8} \cdot \frac{5}{8} = \frac{15}{32} \). The chance of having drawn without replacement is therefore \( \frac{15/28}{15/28+15/32} = \frac{32}{60} = \frac{8}{15} \).

14. Give an example of each of the following:

(a) A connected graph with no cycles.
   A single vertex will do, or any tree.

(b) A graph where every vertex has degree 3.
   \( K_4, K_{3,3}, \) and the graph of a cube will all do.

(c) A graph with an Euler path but no Euler circuit.
   \( K_{2,3} \) will do, or any connected graph with 2 vertices of odd degree.

(d) A graph with a Hamilton cycle but no Euler path.
   \( K_4, \) or \( K_n \) for any even number \( n \geq 4 \).

(e) A graph with \( \chi(G) = \alpha(G) = \omega(G) = 4 \).
   \( K_4 \) with three lone vertices added to the graph.

(f) A non-planar triangle-free graph.
   \( K_{3,3} \).

15. Remove an edge of your choice from \( K_5 \). How many automorphisms does the resulting graph have?

There are 2 vertices of degree 3 and 3 of degree 4. The 2 vertices of degree 3 can be interchanged and the ones of degree 4 can be permuted in any order. The number of automorphisms is therefore \( 2 \cdot 3! = 12 \).

16. I glue triangles and squares together in the shape of a ball so that 4 shapes fit together at every vertex. Show that the number of triangles needed is the same no matter how many squares are used.

Each triangle contributes \( 3/4 - 3/2 + 1 = 1/4 \) and each square contributes \( 4/4 - 4/2 + 1 = 0 \). If \( T \) is the number of triangles and \( S \) is the number of squares, then \( T/4 + 0 \cdot S = 2 \), so \( T = 8 \). One such shape is the octahedron, with 8 triangles and no squares.