Practice Final

Final: Friday, August 14

NAME: ____________________________

1. Show that the expression $(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$ is neither a tautology nor a contradiction.

   If p is false but q is true then the entire expression is $(\mathbf{F} \Rightarrow \mathbf{T}) \Rightarrow (\mathbf{T} \Rightarrow \mathbf{F}) \equiv \mathbf{T} \Rightarrow \mathbf{F} \equiv \mathbf{F}$. Otherwise the entire expression is true.

2. State the negation and prove or disprove: $(\forall x)(\exists y)(\forall z)(xy \geq z)$

   The negation: $(\exists x)(\forall y)(\exists z)(xy < z)$. The original statement is false, and we will show it by proving the negation: let $x = 0$. Then for any y, let $z = 1$. Then $xy = 0 < 1 = z$.

3. Prove that if x and y are positive then $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$.

   Work backwards from the conclusion:

$$\begin{aligned}\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2} &\Leftrightarrow \frac{x^2+y^2}{2} \geq \frac{x^2+2xy+y^2}{4} \\ &\Leftrightarrow 2x^2+2y^2 \geq x^2+2xy+y^2 \\ &\Leftrightarrow x^2-2xy+y^2 \geq 0 \\ &\Leftrightarrow (x-y)^2 \geq 0\end{aligned}$$

   The last inequality is true for all x and y and the inequalities are connected as if-and-only-if statements, so the inequality that we desired to prove is also true.

4. Evaluate: $\sum_{i=1}^{10} \sum_{j=1}^{i} i - 2j$

$$\begin{aligned}\sum_{i=1}^{10}\sum_{j=1}^{i} i-2j &= \sum_{i=1}^{10}\left(i^2 - 2\sum_{j=1}^{i} j\right) \\ &= \sum_{i=1}^{10}\left(i^2 - 2\frac{i^2+i}{2}\right) \\ &= \sum_{i=1}^{10} -i \\ &= -55\end{aligned}$$

5. Find integers $x, y \in \mathbb{Z}$ such that $18x + 40y = 14$

   (a) Method 1: Find the gcd and solve $18x + 40y = 2$, then multiply by 7:

$$\begin{aligned}40 - 2\cdot 18 &= 4 \\ 18 - 4\cdot 4 &= 2 \\ 18 - 4\cdot(40 - 2\cdot 18) &= 2 \\ 9\cdot 18 - 4\cdot 40 &= 2 \\ 63\cdot 18 - 28\cdot 40 &= 14\end{aligned}$$

(b) Method 2: Trial and error. $40-2\cdot 18=4$ and $18-4=14$, so $18-(40-2\cdot 18)=3\cdot 18-40=14$. Note also that $20\cdot 18-9\cdot 40=0$, so the two solutions (63,-28) and (3, -1) differ by a multiple of (20,-9).

6. Determine whether each of the systems of equations has a solution:

   (a)

$$\begin{aligned} x &\equiv 15 \pmod{35} \\ x &\equiv 8 \pmod{10} \\ x &\equiv 1 \pmod{7} \end{aligned}$$

   The first congruence implies that $x\equiv 0 \pmod 5$ and the second that $x \equiv 3 \pmod 5$, so there is no solution.

   (b)

$$\begin{aligned} x &\equiv 3 \pmod{6} \\ x &\equiv 7 \pmod{8} \\ x &\equiv 4 \pmod{5} \end{aligned}$$

   The first two congruences are the only ones involving non-relatively-prime numbers. The first implies that $x\equiv 1 \pmod 2$ and the second that $x\equiv 1 \pmod 2$, and so are compatible. By the Chinese Remainder Theorem, the solution will be unique modulo $3\cdot 8\cdot 5=120$.

7. Prove using induction that if G is a tree with at least 2 vertices then $\chi(G)=2$. You may use the fact that every tree with 2 or more vertices has at least 2 vertices of degree 1.

   Base case: If G has 2 vertices then $\chi(G)=2$.

   Inductive step: suppose that any tree with n vertices can be colored, and consider a tree with $n+1$ vertices. At least one of these vertices has degree 1, so remove it for now. What remains is a tree with n vertices, which by the inductive hypothesis can be colored with 2 colors. Now replace the vertex that was removed, and color it the color opposite its lone neighbor.

8. State the inverse, converse, and contrapositve, and prove or disprove each one: "If a number is divisible by 4 and 5 then it is divisible by 20."

   Forward statement: true. If $4|n$ and $5|n$, then the fact that $\gcd(4,5)=1$ implies that $4\cdot 5|n$.

   Inverse: If a number is not divisible by 4 *or* not divisible by 5 then it is not divisible by 20. True because the inverse is equivalent to the converse.

   Converse: If $20|n$ then $4|n$ and $5|n$. True because if $20|n$ then $20k=n$ for some k, but $20k=5(4k)=4(5k)$.

9. I draw cards from a deck until I have drawn all 4 aces. What is the expected number of kings that I will have drawn?

   Let E_i be the event that the i-th king comes before the last ace. The chance of this is 4/5, since there is a 1/5 chance that a random shuffle of the cards AAAAK will end in the king.

   There are 4 kings, so the total expected number of kings drawn will be $4\cdot(4/5)=16/5$.

10. Prove that if the events E and F are positively correlated then the events E and $\overline{F}$ are negatively correlated.

    We will do a proof by contradiction: say that $p(E\cap F)>p(E)p(F)$ but $p(E\cap\overline{F})\geq p(E)p(\overline{F})$. Then

$$p(E)=p(E\cap(F\cup\overline{F}))=p((E\cap F)\cup(E\cap\overline{F}))=p(E\cap F)+p(E\cap\overline{F})>p(E)p(F)+p(E)p(\overline{F})=p(E).$$

This is a contradiction, so our assumption that $p(E \cap \overline{F}) \geq p(E)p(\overline{F})$ must have been incorrect. Therefore if E and F are positively correlated then E and $\overline{F}$ are negatively correlated.

11. 10 cows, 10 ducks, and 10 pigs are all standing in a line, their positions distributed at random. What is the expected number of times a cow will be standing directly in front of a duck?

    Let $E_{i,j}$ be the event that cow i is directly in front of duck j. The probability of each such event is $29/(30 \cdot 29) = 1/30$. There are 100 such events so the total expected value is $100/30 = 10/3$.

12. If I flip a fair coin 40 times, prove that the probability of getting 30 or more heads is less than or equal to $1/20$.

    The variance is $np(1-p) = 40/4 = 10$, so by Chebyshev's inequality $p(|H - 25| >= 10) = 10/10^2 = 1/10$. This is the probability of getting at least 30 or at most 20 heads, so the probability of getting just 30 or more heads is half that: $1/20$.

13. There is an urn with 5 red balls and 3 yellow balls. I draw 2 balls from the urn, flipping a fair coin to decide whether to draw with or without replacement. If I draw 1 red ball and 1 yellow, what is the probability that I drew without replacement?

    If I draw without replacement then the chance of getting 1 red and 1 yellow is $15/\binom{8}{2} = 15/28$. If I draw with replacement the chance is $2(3/8)(5/8) = 15/32$. The chance of having drawn without replacement is therefore $\frac{15/28}{15/28+15/32} = 32/60 = 8/15$.

14. Give an example of each of the following:

    (a) A connected graph with no cycles.

    A single vertex will do, or any tree.

    (b) A graph where every vertex has degree 3.

    K_4, $K_{3,3}$, and the graph of a cube will all do.

    (c) A graph with an Euler path but no Euler circuit.

    $K_{2,3}$ will do, or any connected graph with 2 vertices of odd degree.

    (d) A graph with a Hamilton cycle but no Euler path.

    K_4, or K_n for any even number $n \geq 4$.

    (e) A graph with $\chi(G) = \alpha(G) = \omega(G) = 4$.

    K_4 with three lone vertices added to the graph.

    (f) A non-planar triangle-free graph.

    $K_{3,3}$.

15. Remove an edge of your choice from K_5. How many automorphisms does the resulting graph have?

    There are 2 vertices of degree 3 and 3 of degree 4. The 2 vertices of degree 3 can be interchanged and the ones of degree 4 can be permuted in any order. The number of automorphisms is therefore $2 \cdot 3! = 12$.

16. I glue triangles and squares together in the shape of a ball so that 4 shapes fit together at every vertex. Show that the number of triangles needed is the same no matter how many squares are used.

    Each triangle contributes $3/4 - 3/2 + 1 = 1/4$ and each square contributes $4/4 - 4/2 + 1 = 0$. If T is the number of triangles and S is the number of squares, then $T/4 + 0 \cdot S = 2$, so $T = 8$. One such shape is the octahedron, with 8 triangles and no squares.