Math 55: Midterm 2
Friday, July 17

NAME: _____

1. (2 points each) Evaluate:

   (a) $\displaystyle\prod_{i=1}^{3}\sum_{j=1}^{2}(j+1) = 5\cdot 5\cdot 5 = 25$

   (b) $\displaystyle\sum_{\substack{d\geq 0 \\ d|15}} d = 1+3+5+15 = 24$

   (c) $\displaystyle\sum_{1\leq i\leq j\leq 3} j = 1+2+3+2+3+3 = 14$

2. (1 point each) Compute:

   (a) $157\cdot 15 \mod 7 = 3\cdot 1 \mod 7 = 3$

   (b) $369\cdot 377 \mod 373 = (-4)\cdot 4 \mod 373 = -16 \mod 373 = 357$

   (c) $2^{364} \mod 7 = (2^6)^{6}0\cdot 2^4 \mod 7 = 2^4 \mod 7 = 1$

   (d) $38^{38} \mod 3 = (-1)^{38} \mod 3 = 1$

3. True or False: $(\exists x \in \mathbb{Z})(\forall y, z \in \mathbb{Z})(13y + 40z \neq x)$. (1 point for answer, 2 for explanation)

   False. Since $\gcd(13, 40) = 1$ there is a linear combination $13a + 40b = 1$, so for any $x \in \mathbb{Z}$ we have the solution $13ax + 40bx = x$, so $y = ax, z = bx$.

4. (3, 3, and 2 points)

   (a) Use the Euclidean Algorithm to find the greatest common divisor of 120 and 35.

   $$120 - 3\cdot 35 = 15$$
   $$35 - 2\cdot 15 = 5$$

   (b) Find any two integers x and y so that $120x + 35y = \gcd(120, 35)$.

   $$35 - 2\cdot (120 - 3\cdot 35) = 5$$
   $$7\cdot 35 - 2\cdot 120 = 5$$

   (c) Find the least common multiple of 120 and 35. The gcd is 5 so the lcm is $120\cdot 35/7 = 120\cdot 7 = 840$.

5. (5 points) Prove that the following system of congruences has no integer solution:

   $x \equiv 5 \pmod{30}$
   $x \equiv 11 \pmod{12}$
   $x \equiv 7 \pmod{15}$

   The first congruence implies $x \equiv 0 \pmod 5$ but the third implies $x \equiv 2 \pmod 5$.

6. Define the Fibonacci sequence by $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$.

   Define the Lucas sequence by $l_0 = 2, l_1 = 1$, and $l_{n+1} = l_n + l_{n-1}$ for $n \geq 1$.

   (a) (2 points) Find l_6. 2, 1, 3, 4, 7, 11, 18. $l_6 = 18$.

   (b) (6 points) Prove that $l_n + l_{n+2} = 5f_{n+1}$ for all $n \geq 0$.

   Proof by induction. Base case: when $n = 0$, $l_0 + l_2 = 2 + 3 = 5 = 5 \cdot f_1$.

   We also need a second base case, since our recurrence relation relies on the previous two terms: when $n = 1, l_1 + l_3 = 1 + 4 = 5 = 5 \cdot f_2$.

   Inductive step: suppose that $l_n + l_{n+2} = 5 \cdot f_{n+1}$. Then

   $$
   \begin{aligned}
   l_{n+1} + l_{n+3} &= l_n + l_{n-1} + l_{n+2} + l_{n+1} \\
   &= (l_n + l_{n+2}) + (l_{n-1} + l_{n+1}) \\
   &= 5f_{n+1} + 5f_n \\
   &= 5f_{n+2}
   \end{aligned}
   $$

7. (6 points) Prove that if $a|m$ and $b|n$ then $ab|mn$.

   If $a|m$ and $b|n$ then $ak = m$ and $bj = n$ for some $j, k \in \mathbb{Z}$. Then $kjab = mn$, so $ab|mn$.

8. (4 points) Find all solutions to $x^2 + 4x \equiv 15 \pmod{19}$ with $0 \leq x < 19$.

   $$
   \begin{aligned}
   x^2 + 4x &\equiv 15 \pmod{19} \\
   x^2 + 4x + 4 &\equiv 0 \pmod{19} \\
   (x + 2)^2 &\equiv 0 \pmod{19} \\
   x + 2 &\equiv 0 \pmod{19} \\
   x &\equiv 17 \pmod{19}
   \end{aligned}
   $$

9. (3 points) Find all solutions to $x^2 \equiv 35 \pmod{65}$ with $0 \leq x < 65$.

   $$
   \begin{aligned}
   x^2 &\equiv 0 \pmod{5} \\
   x &\equiv 0 \pmod{5} \\
   x^2 &\equiv 9 \pmod{13} \\
   x &\equiv \pm 3 \pmod{13} \\
   x &\equiv 10, 55 \pmod{65}
   \end{aligned}
   $$

   Alternatively, say $x^2 \equiv 35 + 65 = 100 \pmod{65}$, so $x \equiv \pm 10 \pmod{65}$.

10. (3 points) Let $a = 5k + 8$ and let $b = 4k + 3$ for some integer k. Show that $\gcd(a, b)$ is either 1 or 17.

    $\gcd(a, b) = \gcd(5k + 8, 4k + 3) = \gcd(k + 5, 4k + 3) = \gcd(k + 5, -17)$, which must be either 1 or 17 since 17 is prime.

    Alternatively: $4a - 5b = (20k + 32) - (20k + 15) = 17$, which implies that $\gcd(a, b)|17$. Since 17 is prime, the gcd must be 1 or 17.