Math 55: Midterm 1

Thursday, July 2

1. (1 point each) True or false? You do not have to show your work.

   (a) $(p \wedge \neg p) \equiv \mathbf{F}$: TRUE

   (b) $(p \wedge q) \Rightarrow p$: TRUE

   (c) $(p \wedge q) \equiv p$: FALSE

   (d) $\mathbb{R} \cup \mathbb{Z} = \mathbb{Z}$: FALSE

   (e) $\mathbb{R} \cap \mathbb{Z} = \mathbb{Z}$: TRUE

2. (4 points) Prove using truth tables that $p \Rightarrow (p \vee q)$ is a tautology.

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

3. (1 point each) Let $A = \{1, 2, 3\}$, let $B = \{2, 4\}$, and let $C = \{1, 2, 5\}$. Find the following:

   (a) $A \cup C = \{1, 2, 3, 5\}$

   (b) $(B - A) \cap C = \emptyset$

   (c) $B \times B = \{(2, 2), (2, 4), (4, 2), (4, 4)\}$

   (d) $\mathcal{P}(B) = \{\emptyset, \{2\}, \{4\}, \{2, 4\}\}$

   (e) $\max(A - C) = 3$

4. (4 points) Illustrate with a Venn Diagram, but do not prove: If $A \subset B$ and $C \subset \overline{B}$ then $A \cap C = \emptyset$.

5. (2 points each) Let f be the proposition "We go to the farmer's market," d be "We go to the deli", m be "We have money," and c be "We buy cheese." Write the following using f, d, m, c, and logical connectives:

   (a) If we go to the farmer's market and have money, we will buy cheese.

   $$(f \wedge m) \Rightarrow c$$

   (b) We will go to the deli if and only if we do not go to the farmer's market.

   $$d \Leftrightarrow \neg f$$

   (c) We cannot buy cheese if we do not have money.

   $$\neg m \Rightarrow \neg c, \text{ or } c \Rightarrow m$$

6. (2 points each) Assume the three statements in the previous question are all true. Which of the following conclusions must be true? Circle all that apply. You **do not** need to show your reasoning.

   (a) We will go to the deli or we will go to the farmer's market. TRUE
   (b) If we did not buy cheese, then we did not have money. FALSE
   (c) If we go to the deli, then we will not buy cheese. FALSE
   (d) If we do not go to the deli, then we will buy cheese if and only if we have money. TRUE

7. (3 points each) Write using quantifier notation. Say whether each statement is true or false. You **do not** have to explain your reasoning.

   (a) For every real number x there is a real number y such that $x + y$ is positive. TRUE

   $$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y > 0)$$

   (b) There is a real number x so that $x + y$ is positive for every real number y. FALSE

   $$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y > 0)$$

8. (6 points) Write the inverse, converse, and contrapositive to this statement: "If $a > 0$ and $b > 0$ then $a \cdot b > 0$." For each of those three statements, say whether it is true (you do not need to prove it) or find a counterexample if it is false.

   - Inverse: If $a \leq 0$ or $b \leq 0$ then $a \cdot b \leq 0$. FALSE: $a = b = -1$.
   - Converse: If $a \cdot b > 0$ then $a > 0$ and $b > 0$. FALSE: $a = b = -1$.
   - If $a \cdot b \leq 0$ then $a \leq 0$ or $b \leq 0$. TRUE

9. (6 points) Pick ONE of the following problems to solve. Circle the problem you intend to solve.

   (a) Let a and b be integers such that $a^2 = b + 1$. Prove that a is even if and only if b is odd.
   (b) Prove that $a^2 + 3a$ is even for any integer a.

   (a) If $a = 2k$ is even then $4k^2 = b + 1$, so $b = 4k^2 - 1 = 2(2k^2 - 1) + 1$ and so b is odd. If $a = 2k + 1$ is odd then $4k^2 + 4k + 1 = b + 1$ so $b = 4k^2 + 4k = 2(2k^2 + 2k)$ and so b is even.
   (b) If $a = 2k$ is even then $a^2 + 3a = 4k^2 + 6k = 2(2k^2 + 3k)$, and so $a^2 + 3a$ is even. If $a = 2k + 1$ is odd then $a^2 + 3a = 4k^2 + 4k + 1 + 6k + 3 = 4k^2 + 10k + 4 = 2(2k^2 + 5k + 2)$, and so $a^2 + 3a$ is even.