Math 55: Homework 6
Due Monday, July 13

1. Use the Extended Euclidean Algorithm to find a solution to $108a + 19b = 1$.

$$
108 - 5 \cdot 19 = 13 \\
19 - 13 = 6 \\
13 - 2 \cdot 6 = 1 \\
13 - 2 \cdot (19 - 13) = 1 \\
3 \cdot 13 - 2 \cdot 19 = 1 \\
3 \cdot (108 - 5 \cdot 19) - 2 \cdot 19 = 1 \\
3 \cdot 108 - 17 \cdot 19 = 1 
$$

2. An army has between 2000 and 3000 soldiers. When they stand in lines of 108 there are 21 left over. When they stand in lines of 19 there are 16 left over. How many soldiers are in the army?

Write out the solution in the form $N = 108a + 19b$. We want to solve $19b \equiv 21 \pmod{108}$ and $108a \equiv 16 \pmod{19}$.

$$
3 \cdot 108 - 17 \cdot 19 = 1 \\
3 \cdot 108 \equiv 1 \pmod{19} \\
-9 \cdot 108 \equiv -3 \equiv 16 \pmod{19} \\
-17 \cdot 19 \equiv 1 \pmod{108} \\
21 \cdot (-17) \cdot 19 \equiv 21 \pmod{108}
$$

So our solutions are $a = -9, b = -357$. Adding 19 to $a$ and 432 to $b$ gives $a = 10, b = 75$. Then $N = 10 \cdot 108 + 75 \cdot 19 = 1080 + 1425 = 2505$.

Any other answer would have to differ by a multiple of $108 \cdot 19 = 2052$, so 2505 is the unique answer in the given range.

3. Let $d = \gcd(a, b)$. Use Bezout’s Theorem to prove that if $e|a$ and $e|b$ then $e|d$.

There are some $s, t$ such that $as + bt = d$. If $e|a$ and $e|b$ then $e|as$ and $e|bt$, so $e|(as + bt)$ and $e|d$.

4. Let $a$ and $b$ be relatively prime. Prove that if $a|n$ and $b|n$ then $ab|n$. (Hint: if $\gcd(a, b) = 1$ and $a|bc$ then $a|c$)

If $a|n$ then $n = ak$ for some integer $k$. Then $b|n$, so $b|ak$. Since $\gcd(a, b) = 1$, $b|k$. Then $k = bc$ for some $c$, meaning that $n = ak = a(bc) = (ab)c$. Therefore $ab|n$. 

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5. Use the result from the previous question: How can you tell if a number is divisible by 45 without doing long division?

If the sum of the digits is divisible by 9 and the number ends in a 0 or a 5— that is, if it is divisible by 9 and 5— then the number is divisible by 45.