Math 55: Final Exam

Friday, August 14

NAME: ______________________________

1. (10 points) Let n be an integer. Prove that $n^2 - 1$ is divisible by 8 if and only if n is odd.

2. Express the following in sum/product notation (6 points), and evaluate (4 points):

$$(2+3+4+5)+(3+4+5+6)+(4+5+6+7)+\cdots+(11+12+13+14)$$

3. Consider the statement "Any integer can be written in the form 21a + 33b, where a and b are integers."

    (a) (3 points each) Write this statement and its negation using quantifier notation.

    (b) (4 points) Prove or disprove the original statement.

4. (10 points) Let $E, F \subset S$ be events such that $p(E) > 0$, $p(F) < 1$, and $E \subset F$. Prove that E and F are positively correlated.

5. (10 points) A week has 7 days. Alice picks 3 days out of the week at random to go to the gym and Bob picks 4 days out of the week at random (and independent of Alice's choices) to go to the gym. Let X be the number of times that Alice and Bob go to the gym on the same day. What is $E(X)$?

6. I have a coin with a 1/4 chance of landing on heads and a 3/4 chance of landing on tails. I flip the coin 100 times.

   (a) (3 points) What is the expected number of heads?

   (b) (4 points) What is the chance that I get exactly 35 heads?

   (c) (3 points) Use Chebyshev's inequality to put an upper bound on the probability of getting 35 or more heads.

7. (10 points) Define the Fibonacci sequence by $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$. Prove that $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for all $n \geq 1$.

8. Give an example of each of the following:

   (a) (3 points) A graph with 3 connected components and 5 edges.

   (b) (3 points) A triangle-free graph with $\chi(G) = 3$.

   (c) (4 points) A graph with an Euler path but no Euler circuit.

9. (5 points each) For each pair of graphs, find an isomorphism between the two graphs or prove that none exists.

10. (10 points) Let $G = (V, E)$ be a graph such that for any pair of vertices $u, v \in V$ there is a unique path connecting u and v. Prove that G is a tree.