Chapter 10.4
Thursday, Week 7

Warmup

What is the diameter of C_7?

How many triangles (3-cycles) are there in K_4?

How many paths are there between any two vertices in C_5?

Is there a connected graph with 5 vertices and 3 edges? How many?

Is there a connected graph with 5 vertices and 4 edges? How many?

Trees

Draw a tree of your choice. How many vertices and edges does it have?

Draw more trees. Count the edges and vertices, and come up with a hypothesis.

Theorems

Theorem 0.1 *If G is a tree, then there is a unique path between any 2 vertices in G.*

Proof: Assume there are 2 paths, then get a contradiction.

Theorem 0.2 *Every tree with at least 2 vertices has at least 2 vertices of degree 1.*

Proof: 1) Look at the endpoints of the longest path.
2) Every graph where every vertex has degree ≥ 2 has a cycle...

Theorem 0.3 *Let G be a graph with n vertices. Then the following statements are equivalent:*

1. *G is a tree (connected and no cycles).*
2. *G has $(n-1)$ edges and no cycles.*
3. *G has $(n-1)$ edges and is connected.*

Proof:

1. (1) $\Rightarrow$ (2): Induction on n, using Theorem 0.2.

2. (2) $\Rightarrow$ (3): Let T be the number of connected components. If component i has v_i vertices, then it has $(v_i - 1)$ edges, thus the whole graph has $(n - T)$ edges. But the whole tree has $(n - 1)$ edges, so $T = 1$.

3. (3) $\Rightarrow$ (1): Suppose it has a cycle, then we could remove an edge without disconnecting the graph. Repeat until there are no cycles, but then there should be $(n - 1)$ edges. Contradiction.