Wednesday, Week 3
Delta-Epsilon Special

Warmup

Find sequences $\{a_n\}$ that satisfy each of the following conditions:

1. $\{a_n\}$ is bounded, but $\lim_{n\to\infty} a_n$ does not exist.
2. $\lim_{n\to\infty} a_n = 0$, but a_n is never equal to zero.
3. $a_n = 0$ infinitely often, but $\{a_n\}$ is not bounded. (Try defining the function separately for even and odd numbers.)

Sequences Described by One Quantifier

- All-zero sequence: $(\forall n)(a_n = 0)$
- "Tiny" sequence: $(\forall n)(|a_n| < 1)$
- "Sometimes zero" sequence: $(\exists n)(a_n = 0)$

Sequences Described by Two Quantifiers

- Constant sequence: $(\exists c)(\forall n)(a_n = c)$
- Bounded sequence: $(\exists M)(\forall n)(|a_n| < M)$
- "Eventually tiny" sequence: $(\exists N)(\forall n > N)(|a_n| < 1)$

- "Eventually zero" sequence: $(\exists N)(\forall n > N)(a_n = 0)$

Three Quantifiers

- $\lim_{n\to\infty} a_n = 0$: $(\forall \epsilon > 0)(\exists N)(\forall n > N)(|a_n| < \epsilon)$

- $\lim_{n\to\infty} a_n \neq 0$: $(\exists \epsilon < 0)(\forall N)(\exists n > N)(|a_n| \geq \epsilon)$

Proof List

1. The sum of two all-zero sequences is all-zero.
2. The product of two "tiny" sequences is tiny.
3. The sum of two tiny sequences is not necessarily tiny.
4. The product of two "sometimes zero" sequences is sometimes zero.
5. The sum and product of two constant/bounded sequences is constant/bounded.
6. All constant sequences are bounded.
7. Tiny sequences are not necessarily constant.
8. The product of two eventually tiny sequences is eventually tiny.
9. The sum and product of two eventually zero sequences are eventually zero.
10. The sum of an eventually tiny sequence and an eventually zero sequence is eventually tiny.
11. Eventually tiny and eventually zero sequences are both bounded.
12. If $\{a_n\}$ is the all-zero sequence then $\lim_{n\to\infty} a_n = 0$.
13. If $\{a_n\}$ is eventually zero then $\lim_{n\to\infty} a_n = 0$.
14. If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$ then $\lim_{n\to\infty}(a_n + b_n) = 0$.
15. If $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded then $\lim_{n\to\infty}(a_n b_n) = 0$.