14.8-15.1: Optimization, Double Integrals

Wednesday, March 16

Optimization

Find the extreme values of $f(x,y) := x^2 + y^2 + 4x - 4y$ on the region $x^2 + y^2 \leq 9$. Sketch the region and a contour plot of f.

$\nabla f(x,y) = \langle 2x+4, 2y-4 \rangle$, so setting the gradient to zero shows that the only critical point is at $(-2,2)$ and the second derivative test shows that this point is a local (therefore the global) min.
Then the maximum must lie on the curve $x^2 + y^2 = 9$, so use Lagrange multipliers: $\langle 2x+4, 2y-4 \rangle = \lambda \rangle 2x, 2y \rangle$ and $x^2 + y^2 = 9$. Combining the first two equations gives $y = -x$, so critical points occur at $(x,y) = (\pm 3/2, \mp 3/2)$. The maximum is at $(3/2, -3/2)$ and the other point is just a minimum on the boundary, not on the whole region.
The region is a disk and the contours are perfect circles, so our answer fits with the intuitive answer that the maximum on the disk occurs at the point farthest from the global min $(-2,2)$.

You are in charge of buying advertising time for a senatorial campaign. Your very scientific models predict that t hours of advertising time in district A will win you $100\sqrt{t}$ new voters and t hours in district B will win you $400\sqrt{t}$ new voters. If the networks in A charge 10 dollars per hour and the networks in B charge 20 dollars per hour and you have 90 dollars to spend, how should you divide your money?

If a is the money devoted to district A and b is the amount spent in district B then you want to maximize $100\sqrt{a} + 400\sqrt{b}$ under the contstraint $10a + 20b = 90$ (assuming you spend all of the money). Using Lagrange multipliers gives the constraints $50/\sqrt{a} = 10\lambda, 200/\sqrt{b} = 20\lambda$, so $100/\sqrt{a} = 200/\sqrt{b}$ and therefore $b = 4a$. Combining this with the constraint $a + 2b = 9$ gives $a = 1, b = 4$, so you should spend 10 dollars in district A and 80 in district B.

Optimization with Two Constraints

Find the maximum and minimum values of $f(x,y,z) = x+y+z$ given the constraints $x^2+y^2+z^2 = 1, x = 2y$.
$\nabla f(x,y,z) = \langle 1,1,1 \rangle$, so setting $g(x,y,z) = x^2 + y^2 + z^2$ and $h(x,y,z) = x - 2y$ and using Lagrange multipliers gives

$$\begin{aligned}
\langle 1,1,1 \rangle &= \lambda \langle x,y,z \rangle + \mu \langle 1,-2,0 \rangle \\
1 &= \lambda x + \mu \\
1 &= \lambda y - 2\mu \\
1 &= \lambda z \\
x^2 + y^2 + z^2 &= 1 \\
x &= 2y.
\end{aligned}$$

Eliminating μ from the first two equations gives $3 = \lambda y + 2\lambda x$, and substituting $x = 2y$ gives $\lambda y = 3/5$. Since $\lambda z = 1$, it follows that $y = 3z/5$, and so $x = 6z/5$. Therefore $((6/5)^2 + (3/5)^2 + 1)z^2 = 1$, and $z = \pm\sqrt{5/14}$, and x and y follow. The point where $x,y,z > 0$ is the maximum and the one where $x,y,z < 0$ is the minimum.

Double Integrals!

Sketch the solid whose volume is given by the integral $\int_0^1 \int_0^1 (4 - x - 2y)\, dx\, dy$ and find the volume.

$\int_0^1 \int_0^1 (4 - 2x - 2y)\, dx\, dy = \int_0^1 3 - 2y\, dy = 2.$

Find the integral $\iint_R ye^{-xy}\, dA$ on the region $R = [0, 2] \times [0, 3]$.

$\iint_R ye^{-xy}\, dA = \int_{y=0}^3 \int_{x=0}^2 ye^{-xy}\, dx\, dy = \int_{y=0}^3 (-e^{-xy})|0^2\, dy = \int_0^3 1 - e^{-2y}\, dy = y + \frac{1}{2}e^{-2y}|0^3 = 3 + (e^{-6} - 1)/2 = 5/2 + e^{-6}/2.$