14.6-7: Gradients and Critical Points

Wednesday, March 9

Gradients

The temperature at a point \((x, y, z)\) is given by \(T(x, y, z) = 200e^{-x^2 - 3y^2 - 9z^2}\) where \(T\) is in Celsius and \(x, y, z\) in meters.

1. Find the rate of change in temperature at the point \(P(2, -1, 2)\) in the direction toward the point \((3, -3, 3)\).
   
   \[
   \langle P_x, P_y, P_z \rangle = \langle -2x, -6y, -18z \rangle \cdot 200e^{-x^2 - 3y^2 - 9z^2}, \text{ so in the direction (1, -2, 1) the rate of change is } 200e^{-2^2 - 3(-1)^2 - 9(2)^2}(4, 6, -36) \cdot \langle 1, -2, 1 \rangle / \sqrt{27} = 200e^{-43}(-52) / \sqrt{6}. \text{ This will be a fairly small change since } e^{-43} \text{ is a very small number.}
   \]

2. In which direction does the temperature increase fastest at \(P\)?
   
   The rate of fastest increase is in the direction of the gradient, so in the direction \((-4, 6, -36)\).

3. Find the maximum rate of temperature increase at \(P\).
   
   \[\nabla P(2, -1, 2) \cdot \mathbf{v} / |\mathbf{v}|, \text{ where } \mathbf{v} \text{ is the answer from the previous part.}\]

   If \(L(x, y)\) is the linear approximation to a function \(f(x, y)\) at a point \((x_0, y_0)\), express \(L\) in terms of \(\nabla f(x_0, y_0)\).

   \[L(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0).\]

   Also express the Chain rule in terms of the gradient.

Example: For the function \(H(x, y)\) with \(x = f(t), y = g(t)\), \(dH/dt = \nabla H(x, y) \cdot (f'(t), g'(t)).\)

Or if \(r = \langle x, y \rangle\), then \(dH/dt = \nabla H(r) \cdot r'(t): \text{ the derivative is the dot product of the gradient and the velocity vector.}\)

Critical Points

Find all critical points of the following functions. Apply the Second Derivative test at those points, and use the information to sketch the graphs of the functions.

- \(f(x, y) = 2x^2 - 2xy + 5y^2 - 5\)
  \[
  \nabla f(x, y) = \langle 4x - 2y, 10y - 2x \rangle, \text{ which is equal to zero only when } (x, y) = (0, 0). \text{ The second derivatives are } f_{xx} = 4, f_{yy} = 10, f_{xy} = -2 \text{ and } 4 \cdot 10 - (-2)^2 > 0, \text{ so the point } (0, 0) \text{ is a local minimum. The whole graph is an elliptic paraboloid.}
  \]

- \(f(x, y) = x^3 - x - y^2\)
  \[
  \nabla f(x, y) = \langle 3x^2 - 1, 2y \rangle \text{ which is equal to zero at } x = \pm 1/\sqrt{3}, y = 0. \text{ The Second Derivative test indicates a local maximum at } (-1/\sqrt{3}, 0) \text{ and a saddle point at } (1/\sqrt{3}, 0).
  \]

- \(f(x, y) = (x - y)(1 - xy)\)
  
  Rewrite \(f(x, y) = x - y - x^2y + y^2x\). \(\nabla f(x, y) = (1 - 2xy + y^2, -1 - x^2 + 2xy). \text{ Adding the two equations gives } x^2 = y^2 \text{ (so } x = \pm y). \text{ This leads to the two solutions } (1, 1), (-1, -1)\)

  The second partial derivatives are \(f_{xx} = -2y, f_{yy} = 2x, f_{xy} = 2y - 2x\), so the Second Derivative test gives that both points are saddle points.