14.6-7: Gradients and Critical Points

Wednesday, March 9

Gradients

The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$ where T is in Celsius and x, y, z in meters.

1. Find the rate of change in temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

   $\langle P_x, P_y, P_z\rangle = \langle -2x, -6y, -18z\rangle \cdot 200e^{-x^2-3y^2-9z^2}$, so in the direction $(1, -2, 1)$ the rate of change is $200e^{-2^2-3\cdot 1^2-9\cdot 2^2}\langle -4, 6, -36\rangle \cdot \langle 1, -2, 1\rangle/\sqrt{27} = 200e^{-43}(-52)/\sqrt{6}$. This will be a fairly small change since e^{-43} is a very small number.

2. In which direction does the temperature increase fastest at P?

   The rate of fastest increase is in the direction of the gradient, so in the direction $\langle -4, 6, -36\rangle$.

3. Find the maximum rate of temperature increase at P.

   $\nabla P(2, -1, 2) \cdot \mathbf{v}/|\mathbf{v}|$, where $\mathbf{v}$ is the answer from the previous part.

If $L(x, y)$ is the linear approximation to a function $f(x, y)$ at a point (x_0, y_0), express L in terms of $\nabla f(x_0, y_0)$.
$L(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0\rangle$.
Also express the Chain rule in terms of the gradient.
Example: For the function $H(x, y)$ with $x = f(t), y = g(t)$, $dH/dt = \nabla H(x, y) \cdot \langle f'(t), g'(t)\rangle$.
Or if $\mathbf{r} = \langle x, y\rangle$, then $dH/dt = \nabla H(r) \cdot \mathbf{r}'(t)$: the derivative is the dot product of the gradient and the velocity vector.

Critical Points

Find all critical points of the following functions. Apply the Second Derivative test at those points, and use the information to sketch the graphs of the functions.

- $f(x, y) = 2x^2 - 2xy + 5y^2 - 5$

  $\nabla f(x, y) = \langle 4x - 2y, 10y - 2x$, which is equal to zero only when $(x, y) = (0, 0)$. The second derivatives are $f_{xx} = 4, f_{yy} = 10, f_{xy} = -2$ and $4 \cdot 10 - (-2)^2 > 0$, so the point $(0, 0)$ is a local minimum. The whole graph is an elliptic paraboloid.

- $f(x, y) = x^3 - x - y^2$

  $\nabla f(x, y) = \langle 3x^2 - 1, 2y\rangle$ which is equal to zero at $x = \pm 1/\sqrt{3}, y = 0$. The Second Derivative test indicates a local maximum at $(-1/\sqrt{3}, 0)$ and a saddle point at $(1/\sqrt{3}, 0)$.

- $f(x, y) = (x - y)(1 - xy)$

  Rewrite $f(x, y) = x - y - x^2y + y^2x$. $\nabla f(x, y) = \langle 1 - 2xy + y^2, -1 - x^2 + 2xy$. Adding the two equations gives $x^2 = y^2$ (so $x = \pm y$). This leads to the two solutions $(1, 1), (-1, -1)$

  The second partial derivatives are $f_{xx} = -2y, f_{yy} = 2x, f_{xy} = 2y - 2x$, so the Second Derivative test gives that both points are saddle points.