# 13.4,14.1: Functions of Multiple Variables

Wednesday, March 2

## Partial Derivatives, Linear Approximation

Find all of the second partial derivatives of the following functions. In particular, verify that $f_{xy} = f_{yx}$.

1. $f(x, y) = \cos(xy) + xe^y$:

$$\begin{aligned} f_x &= -y\sin(xy) + e^y \\ f_y &= -x\sin(xy) + xe^y \\ f_{xx} &= -y^2\cos(xy) \\ f_{xy} &= -\sin(xy) - xy\cos(xy) + e^y \\ f_{yx} &= -\sin(xy) - xy\cos(xy) + e^y \\ f_{yy} &= -x^2\cos(xy) + xe^y \end{aligned}$$

2. $f(x, y) = \arctan(y/x)$: Just stick with the first partial derivatives since the second partial derivatives are annoying:

$$\begin{aligned} f_x &= \frac{1}{1+(y/x)^2}\frac{-y}{x^2} = \frac{-y}{x^2+y^2} \\ f_y &= \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}. \end{aligned}$$

There's something going on here related to circles... note that $(f_x)^2 = (f_y)^2 = 1/(x^2 + y^2)$ for all $x$ and $y$.

3. $f(x, y) = \sqrt{x^2 + y^2}$

$$\begin{aligned} f_x &= \frac{x}{\sqrt{x^2+y^2}} \\ f_y &= \frac{y}{\sqrt{x^2+y^2}}. \end{aligned}$$

Circles again... this time, $(f_x)^2 + (f_y)^2 = 1$.

Find a linear approximation to the function $f(x, y, z) = 3xy+xz+ye^{x^2z}$ at $(0, 1, 0)$ and use it to approximate $f(0.1, 0.9, 0.2)$.
First find the partial derivatives:

$$\begin{aligned} f_x(0,1,0) &= 3y + z + 2yze^{x^2z} = 3 \\ f_y(0,1,0) &= 3x + e^{x^2z} = 1 \\ f_z(0,1,0) &= x + x^2ye^{x^2z} = 0. \end{aligned}$$

Therefore, $f(.1, .9, .2) \approx f(0, 1, 0) + 3(.1) + 1(-.1) + 0(.2) = 1.2$.

## Even More Projectile Motion

A cannoneer wishes to hit a target on the ground, and to do so fires a cannonball with velocity $v_0$ and at angle of elevation $\theta$. Recall that the distance the cannonball travels through the air is given by $d(v_0, \theta) = \frac{1}{g} v_0^2 \sin 2\theta$.

1. Make (rough) sketches of the graph of $d$ for $0 \leq v \leq \sqrt{g}$ and $0 \leq \theta \leq \pi/2$.

   Three edges of the plot will be zero (if $\theta = 0$ or $\theta = \pi/2$ or $v_0 = 0$, the object will hit the ground immediately), and the fourth will be half of a sine wave with its maximum at $\theta = \pi/4$. For any fixed $\theta$, the function will increase quadratically with respect to $v_0$.

2. Sketch a contour plot of $d$. If the target is a distance of 1/2 (units) from the cannoneer, sketch the curve of values $(v_0, \theta)$ that hit the target.

   To hit a particular distance $k$, we could need $v_0^2 \sin 2\theta = gk$, or $v_0^2 = gk \csc 2\theta$. Roughly, this means that the closer the angle is to 0 or $\pi/2$, the greater the velocity will need to be to have the cannonball travel the same distance. When $\theta = \pi/4$, the required velocity is at a minimum.

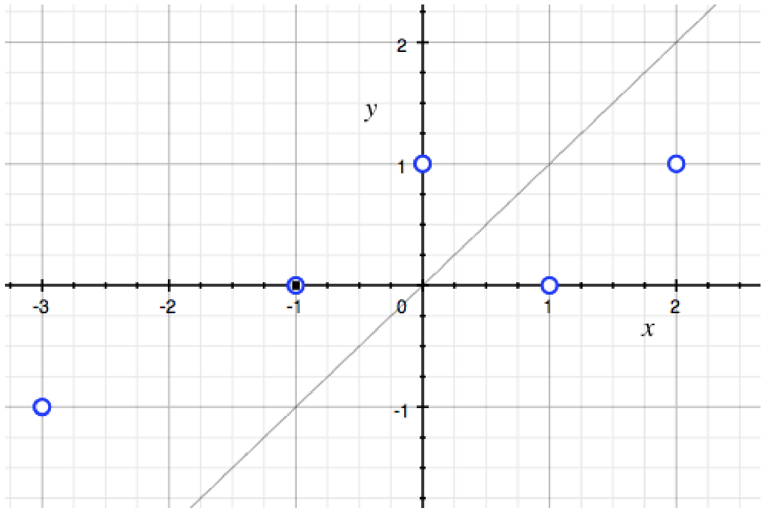
3. The cannoneer does not have perfect accuracy and so will have some small error in both $v_0$ and $\theta$. Use a linearization of $d(v_0, \theta)$ to decide: is it a better idea (accuracywise) to fire the ball at a high angle, a low angle, or a 45-degree angle? Explain.

   The smaller the partial derivatives, the more stable the function is when the inputs are perturbed. $d_\theta = \frac{2}{g} v_0^2 \cos 2\theta$, which is at a maximum when $\theta = 0$ or $\pi/2$ and is zero when $\theta = \pi/4$ (45 degrees). $d_{v_0} = \frac{2}{g} v_0 \sin 2\theta$, so $d$ is more stable the smaller the velocity is (with respect to absolute changes rather than changes proportional to $v_0$).

   When the ball is fired at 45 degrees the required velocity is at a minimum, so with respect to both angle and velocity the best-case scenario is to fire at 45 degrees.

## Best Fit Line

We would like to approximate a set of data points with a linear function. The picture below shows the five points (-3,-1), (0,1), (-1,0), (1,0), and (2,1) approximated by the function $y = x$.




1. We would like to find the values $a$ and $b$ that give the "best fit line" in the form $y = ax + b$. How do you think the current values $a = 1$ and $b = 0$ should be adjusted?

   Maybe $a$ should be smaller and $b$ larger, I guess?

2. We can measure the error of the function by summing the squared distances of our predicted values from the actual y-values of the data points: so $E(a, b) = \sum_i (ax_i + b - y_i)^2$. In our case, the error function for these five data points is $E(a, b) = 15a^2 + 5b^2 + 3 - 2ab - 2b - 10a$. Find $E_a$ and $E_b$.

   $E_a = 30a - 2b - 10$ and $E_b = 10b - 2a - 2$.

3. Use your previous answer: how should $a$ and $b$ be altered to improve the fit of the line?

   At the point $(a, b) = (1, 0)$ (the current line), $E_a = 20$ and $E_b = -22$. This means that (with our linear approximation), the error will go down when $a$ decreases and $b$ increases.

   If we want to find the optimal pair $(a, b)$, we have to solve the system

   $$\begin{aligned} 30a - 2b &= 10 \\ -2a + 10b &= 2, \end{aligned}$$

   which has $a = 13/37, b = 10/37$ as the unique solution. The best fit line is therefore $y = \frac{13}{37}x + \frac{10}{37}$.