13.4,14.1: Functions of Multiple Variables

Wednesday, February 24

Formulas

- $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$
- $\kappa = |d\mathbf{T}|/|ds| = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$

Velocity and Acceleration

A ball attached to a stiff rod swings back and forth. The only forces acting on the ball are gravity (acting with acceleration g straight downward) and the rod itself. Suppose the ball is released from a point where the rod is parallel. Sketch the ball's position and its velocity and acceleration vectors when the rod is

1. Parallel to the ground.

   The acceleration vector is directed straight downward with magnitude g since the tension in the rod is zero. The velocity vector and

2. Vertical.

   Neither the tension nor gravity vectors have any tangential component, so the acceleration vector is pointed straight up (to account for the curvature). The velocity vector is horizontal (always tangent to the semicircular path of the ball) and has a large magnitude since this is the point when the ball is moving the fastest.

3. At a 45 degree angle relative to the ground.

   Gravity has some tangential component making the ball speed up, and the net normal component has to be directed into the circle to account for the curvature. The acceleration vector will point upward and in the direction the ball is moving. (assuming that the ball is falling. If the ball is rising then the tangential component of gravity will be in the direction opposite the ball's velocity.)

4. When is the ball's velocity at a maximum or minimum?

   Velocity is at a minimum (zero) when the rod is horizontal and at a maximum when the rod is vertical.

5. If T is the force exerted by the rod and m is the mass of the ball, show that $|T| \geq mg\cos\theta$ where θ is the angle between the vertical axis and the rod.

   The normal component of the tension T is directed into the circle and the normal component of gravity $mg\cos\theta$ is directed outward.

6. When is the magnitude of the force on the ball at a maximum or minimum? Can you express the magnitude of the force as a function of the ball's speed and the rod angle θ?

   $F = ma$, so the force is at a max or min when the acceleration is. Then we know that $a = |v|'\mathbf{T} + \kappa|v|^2\mathbf{N}$. The first part in this sum is the tangential component of acceleration, which is equal to $g\sin\theta$ (since only gravity has any tangential component to it). Since the path of the ball is a circle (say, of radius r), κ is a constant $1/r$. The acceleration is therefore $a = g\sin\theta\mathbf{T} + |v|^2/r\mathbf{N}$, in terms of $|v|$ and θ.

   In order to find the maximum acceleration, you would probably need to use formulas for the potential and kinetic energy of an object in order to relate θ and $|v|$. Leave it for a physics class.

Contour Plots

There are three points $A(0,0)$, $B(0,1)$, $C(1,0)$ that you would like to live close to. Where should you place your house such that the sum of the *squares* of your distances from A, B, and C is minimized? If $T(x,y)$ is the sum of the three distances from your location (x,y), make a contour plot of T.
For any $X = (x,y)$, we can get

$$\begin{aligned}|X-A|^2 + |X-B|^2 + |X-C|^2 &= (x^2+y^2) + (x^2+(y-1)^2) + ((x-1)^2+y^2)\\ &= 3x^2 - 2x + 3y^2 - 2y + 2.\end{aligned}$$

The cost function $T(x,y)$ is an elliptic paraboloid with minimum at $(1/3, 1/3)$ (the center of mass of triangle ABC!) The contours of the contour plot should be circles with $(1/3, 1/3)$ as their center.

What if you just want to minimize the sum of the distances and not the sum of the squares?
This is much harder.