13.3: Arc Length and Curvature
Wednesday, February 17

Relevant Formulas

- \( T(t) = \frac{r'(t)}{|r'(t)|} \)
- \( B(t) = T(t) \times N(t) \)
- \( N(t) = \frac{T'(t)}{|T'(t)|} \)
- \( \kappa = \frac{|dT|}{ds} = \frac{|T'(t)|}{|r'(t)|} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \)

Sketches

Sketch:

1. \( x^2 - 2x + y^2 = -z^2 \): It’s a sphere of radius 1 centered at (1,0,0).

2. \((x + y)^2 + z^2 = 1\): A cylinder of radius 1 with axis of symmetry \( x + y = 0, z = 0 \).

3. The intersection of the unit sphere and the surface defined by \((2x)^2 + (2y)^2 = 1\): Two circles of radius 1/2 at \( z = \pm \sqrt{3}/2 \).

4. \( r(t) = \langle \sin t, \cos t, t \rangle \): A helix spiraling upward and clockwise around a cylinder.

5. For the previous problem, pick a point and sketch \( T(t), N(t), \text{and} B(t) \). \( T(t) \) is tangent to the cylinder, \( N(t) \) points inward, and \( B(t) \) points downward.
Arc Length and Curvature

1. Given the curve \( \mathbf{r}(t) = \langle 5 - t, 4t - 3, 3t \rangle \), find the point 4 units along the curve from the point \((4, 1, 3)\) as \(t\) increases.

   The position as a function of distance traveled is \( \mathbf{r}(t(s)) = \langle 4 - s/\sqrt{26}, 4s/\sqrt{26}, 3 + 3s/\sqrt{26} \rangle \), so plug in \(s = 4\) to get the point 4 units along the curve.

2. Find formulas for the tangent and normal vectors and the curvature of the curve \( \mathbf{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle \).

\[
\mathbf{r}'(t) = \langle 1, t, 2t \rangle \\
\mathbf{T}(t) = \langle 1, t, 2t \rangle / \sqrt{1 + 5t^2} \\
\mathbf{T}'(t) = \langle -5t(1 + 5t^2)^{-3/2}, (1 + 5t^2)^{-1/2} - 5t^2(1 + 5t^2)^{-3/2}, 2(1 + 5t^2)^{-1/2} - 10t^2(1 + 5t^2)^{-3/2} \rangle \\
= (1 + 5t^2)^{-3/2} \langle -5t, 1, 2 \rangle \\
\mathbf{N}(t) = \frac{\langle -5t, 1, 2 \rangle}{\sqrt{5 + 25t^2}} \\
\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \\
= \frac{\sqrt{5}}{(1 + 5t^2)^{3/2}}.
\]

A mistake in the answer here is entirely possible, but the rough idea is that the particle follows a parabolic shape and so the curvature decreases over time since the path gets closer and closer to a straight line.

3. Find the tangent, normal, and binormal vectors for the curve \( \mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle \) at the point \((1, 2/3, 1)\).

\[
\langle 2/3, 2/3, 1/3 \rangle, \langle -1/3, 2/3, -2/3 \rangle, \langle -2/3, 1/3, 2/3 \rangle
\]

Miscellany

A particle moves around the surface of a sphere. Show that its velocity vector and the vector from the center of the sphere to the particle’s position are orthogonal at all times.

If \( x(t)^2 + y(t)^2 + z(t)^2 = 1 \), then taking the derivative with respect to time gives \(0 = 2xx' + 2yy' + 2zz' = 2(x, y, z) \cdot (x', y', z')\).