13.3: Arc Length and Curvature

Wednesday, February 17

Relevant Formulas

- $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
- $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$
- $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$
- $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$

Sketches

Sketch:

1. $x^2 - 2x + y^2 = -z^2$: It's a sphere of radius 1 centered at (1,0,0).
2. $(x+y)^2 + z^2 = 1$: A cylinder of radius 1 with axis of symmetry $x + y = 0, z = 0$.
3. The intersection of the unit sphere and the surface defined by $(2x)^2 + (2y)^2 = 1$: Two circles of radius 1/2 at $z = \pm\sqrt{3}/2$.
4. $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$: A helix spiraling upward and clockwise around a cylinder.
5. For the previous problem, pick a point and sketch $\mathbf{T}(t), \mathbf{N}(t)$, and $\mathbf{B}(t)$. $\mathbf{T}(t)$ is tangent to the cylinder, $\mathbf{N}(t)$ points inward, and $\mathbf{B}(t)$ points downward.

Arc Length and Curvature

1. Given the curve $\mathbf{r}(t) = \langle 5 - t, 4t - 3, 3t \rangle$, find the point 4 units along the curve from the point $(4, 1, 3)$ as t increases.

   The position as a function of distance traveled is $\mathbf{r}(t(s)) = (4 - s/\sqrt{26}, 4s/\sqrt{26}, 3 + 3s\sqrt{26})$, so plug in $s = 4$ to get the point 4 units along the curve.

2. Find formulas for the tangent and normal vectors and the curvature of the curve $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$.

   $$\begin{aligned}
   \mathbf{r}'(t) &= \langle 1, t, 2t \rangle \\
   \mathbf{T}(t) &= \langle 1, t, 2t \rangle / \sqrt{1 + 5t^2} \\
   \mathbf{T}'(t) &= \langle -5t(1+5t^2)^{-3/2}, (1+5t^2)^{-1/2} - 5t^2(1+5t^2)^{-3/2}, 2(1+5t^2)^{-1/2} - 10t^2(1+5t^2)^{-3/2} \rangle \\
   &= (1+5t^2)^{-3/2} \langle -5t, 1, 2 \rangle \\
   \mathbf{N}(t) &= \frac{\langle -5t, 1, 2 \rangle}{\sqrt{5 + 25t^2}} \\
   \kappa(t) &= \frac{|\mathbf{T}'(t)}{|\mathbf{r}'(t)|} \\
   &= \frac{\sqrt{5}}{(1+5t^2)^{3/2}}.
   \end{aligned}$$

   A mistake in the answer here is entirely possible, but the rough idea is that the particle follows a parabolic shape and so the curvature decreases over time since the path gets closer and closer to a straight line.

3. Find the tangent, normal, and binormal vectors for the curve $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$ at the point $(1, 2/3, 1)$.

   $$\langle 2/3, 2/3, 1/3 \rangle, \langle -1/3, 2/3, -2/3 \rangle, \langle -2/3, 1/3, 2/3 \rangle$$

Miscellany

A particle moves around the surface of a sphere. Show that its velocity vector and the vector from the center of the sphere to the particle's position are orthogonal at all times.
If $x(t)^2 + y(t)^2 + z(t)^2 = 1$, then taking the derivative with respect to time gives $0 = 2xx' + 2yy' + 2zz' = 2\langle x, y, z \rangle \cdot \langle x', y', z' \rangle$.