12.5-6: Planes, Cylinders, Quadratic Surfaces

Wednesday, February 10

Warmup

Sketch the graphs of the surfaces described by the following equations:

1. $6x - 3y + 4z = 6$: Plane
2. $x^2 + y^2 = 1$ (in $\mathbb{R}^3$): Cylinder
3. $x^2 + y^2 + z^2 = 1$: Sphere
4. $z = x^2 + 2y^2$: Elliptic paraboloid
5. $z = x^2 - 2y^2$: hyperbolic paraboloid
6. $z^2 = 2x^2 + y^2$: Cone [EDIT: previous posted answer was a typo]

Planes

Which of the following four planes are parallel? Are any of them identical?

$$P_1 : 3x + 6y - 3z = 6, \quad P_2 : 4x - 12y + 8z = 5, \quad P_3 : 9y = 1 + 3x + 6z, \quad P_4 : z = x + 2y - 2$$

The normal vectors to the four planes are $\langle 3, 6, -3\rangle, 4, -12, 8\rangle, \langle 3, -9, 6\rangle, \langle 1, 2, -1\rangle$. The first and fourth are parallel (in fact, identical) and the middle two are parallel (but not identical).

Find an equation for the plane that passes through the points $(2, 1, 2), (3, -8, 6)$, and $(-2, -3, 1)$.
Get the vectors between the first and second points ($\langle 1, -9, 4\rangle$) and between the second and third points ($\langle 5, -5, 5\rangle$), and take their cross product to find a normal line ($\langle -25, 15, 40\rangle$, which can be scaled down to $\langle -5, 3, 8\rangle$). The plane is of the form $\langle -5, 3, 8\rangle \cdot x = k$, for some k.
Plugging in the point $(2, 1, 2)$ gives $k = 9$.

Find an equation for the plane that passes through the point $(3, 1, 4)$ and contains the line of intersection of the planes $x + 2y + 3z = 1$ and $2x - y + z = -3$.
The line of intersection of the two planes is parallel to $\langle 1, 2, 3\rangle \times \langle 2, -1, 1\rangle$, which is $\langle 5, 5, -5\rangle$, which is parallel to $\langle 1, 1, -1\rangle$.
Next we have to find a specific point on this line. Look for one at $z = 0$, and get the two equations $x + 2y = 1, 2x - y = -3$, which has the solution $(-1, 1, 0)$.
The vector from $(-1, 1, 0)$ to $(3, 1, 4)$ is $\langle 4, 0, 4\rangle$, and the plane is parallel to this vector and to $\langle 1, 1, -1\rangle$. So the plane is normal to the cross product, which is $\langle -4, 8, 4\rangle$ (or parallel to $\langle -1, 2, 1\rangle$).
Finally, $\langle -1, 2, 1\rangle \cdot \langle 3, 1, 4\rangle = 3$, so the plane is of the form $\langle -1, 2, 1\rangle \cdot x = 3$.

Conic sections

Consider the cone described by $z^2 = x^2 + y^2$. Sketch its intersection with the following planes. Use substitution to eliminate one of the three variables, and describe the intersection.

1. $z = 5$: circle
2. $x = 2$: hyperbola
3. $z = y + 1$: parabola
4. $z = (x/2) + 1$: ellipse

Find an equation for the surface consisting of all points equidistant from the points $(-1, 0, 0)$ and the plane $x = 1$. Identify the surface.
$(x - 1)^2 = (x + 1)^2 + y^2 + z^2$, or $-4x = x^2 + z^2$. It's an elliptic paraboloid.