12.5-6: Planes, Cylinders, Quadratic Surfaces
Wednesday, February 10

Warmup

Sketch the graphs of the surfaces described by the following equations:

1. \(6x - 3y + 4z = 6\): Plane
2. \(x^2 + y^2 = 1\) (in \(\mathbb{R}^3\)): Cylinder
3. \(x^2 + y^2 + z^2 = 1\): Sphere
4. \(z = x^2 + 2y^2\): Elliptic paraboloid
5. \(z = x^2 - 2y^2\): hyperbolic paraboloid
6. \(z^2 = 2x^2 + y^2\): Cone

Planes

Which of the following four planes are parallel? Are any of them identical?

\(P_1 : 3x + 6y - 3z = 6\), \(P_2 : 4x - 12y + 8z = 5\), \(P_3 : 9y = 1 + 3x + 6z\), \(P_4 : z = x + 2y - 2\)

The normal vectors to the four planes are \(\langle 3, 6, -3 \rangle\), \(\langle 4, -12, 8 \rangle\), \(\langle 3, -9, 6 \rangle\), \(\langle 1, 2, -1 \rangle\). The first and fourth are parallel (in fact, identical) and the middle two are parallel (but not identical).

Find an equation for the plane that passes through the points \((2, 1, 2), (3, -8, 6), (-2, -3, 1)\).

Get the vectors between the first and second points \(\langle 1, -9, 4 \rangle\) and between the second and third points \(\langle 5, -5, 5 \rangle\), and take their cross product to find a normal line \(\langle -25, 15, 40 \rangle\), which can be scaled down to \(\langle -5, 3, 8 \rangle\). The plane is of the form \(\langle -5, 3, 8 \rangle \cdot x = k\), for some \(k\).

Plugging in the point \((2, 1, 2)\) gives \(k = 9\).

Find an equation for the plane that passes through the point \((3, 1, 4)\) and contains the line of intersection of the planes \(x + 2y + 3z = 1\) and \(2x - y + z = -3\).

The line of intersection of the two planes is parallel to \(\langle 1, 2, 3 \rangle \times \langle 2, -1, 1 \rangle\), which is \(\langle 5, 5, -5 \rangle\), which is parallel to \(\langle 1, 1, -1 \rangle\).

Next we have to find a specific point on this line. Look for one at \(z = 0\), and get the two equations \(x + 2y = 1, 2x - y = -3\), which has the solution \((-1, 1, 0)\).

The vector from \((-1, 1, 0)\) to \((3, 1, 4)\) is \(\langle 4, 0, 4 \rangle\), and the plane is parallel to this vector and to \((1, 1, -1)\). So the plane is normal to the cross product, which is \(\langle -4, 8, 4 \rangle\) (or parallel to \(\langle -1, 2, 1 \rangle\)).

Finally, \(\langle -1, 2, 1 \rangle \cdot \langle 3, 1, 4 \rangle = 3\), so the plane is of the form \(\langle -1, 2, 1 \rangle \cdot x = 3\).
Conic sections
Consider the cone described by \( z^2 = x^2 + y^2 \). Sketch its intersection with the following planes. Use substitution to eliminate one of the three variables, and describe the intersection.

1. \( z = 5 \): circle
2. \( x = 2 \): hyperbola
3. \( z = y + 1 \): parabola
4. \( z = (x/2) + 1 \): ellipse

Find an equation for the surface consisting of all points equidistant from the points \((-1, 0, 0)\) and the plane \( x = 1 \). Identify the surface.
\[
(x - 1)^2 = (x + 1)^2 + y^2 + z^2,
\]
or
\[
-4x = x^2 + z^2.
\]
It’s an elliptic paraboloid.