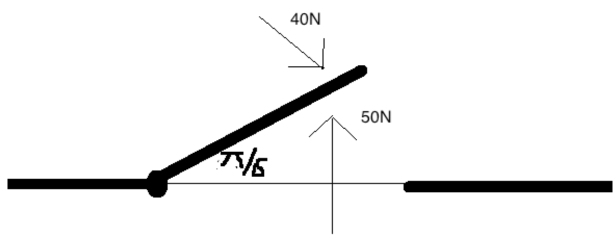
12.3: Dot Product

Wednesday, February 3

How to Open a Door

One student (pushing upward with a force of 50N) is trying to open a door and another (pushing with a force of 40N at an angle of $-\pi/4$) is trying to shut it. Which way will the door start to move?



The door will start to move in a direction perpendicular to the angle it sits at. The forces acting in the relevant direction are the total forces times the cosine of the angle they make with the vector describing the direction the door will move. The relevant forces are therefore $40\cos 15 \approx 38N$ and $50\cos 30 \approx 43N$, so the door will move outward.

Draw vectors a, b, c, d such that $a \cdot b > 0, a \cdot c = 0, a \cdot d < 0$.
a makes acute, right, and obtuse angles with b, c, d, respectively.

Projections

Let $a = \langle 1, 2, 2\rangle, b = \langle 3, -1, 0\rangle$.

1. Find the angle between a and b.

   $\cos\theta = a \cdot b/|a||b| = 1/3\sqrt{10} \approx 84°$. The vectors are close to perpendicular.

2. Find the vector projection of b onto a.

   The projection is $(a \cdot b/|a|^2)a = a/9$.

3. Find a vector orthogonal to a.

   Go by trial and error, setting the first two components and deciding what the third needs to be. $\langle 0, 1, -1\rangle$ will do.

4. Find a vector orthogonal to both a and b.

   Trial and error will again work okay. Say the vector is $v = \langle x, y, z\rangle$ and $x = 1$. Then $v \cdot b = 0$ implies $-y = 3$, so $y = -3$. Then $v \cdot a = 0$ forces $z = 5/2$. An orthogonal vector is $\langle 1, -3, 5/2\rangle$.

   In more complicated scenarios we can use projections, but that may be a topic for another day...

5. Are there any vectors orthogonal to a, b, and the vector from your last answer?

   Nope. Those 3 vectors span the entire space, so nothing can be orthogonal to all of them (only the zero vector has zero mass in each of the 3 components).

Work

A ball of mass m is dropped from height r. If gravity acts downward on the ball with a constant force of mg, how much work does gravity do from the time the ball is dropped until the time it hits the ground?
Gravity does mgr work.
What if the ball is rolled down a plane with a $30°$ incline starting at height r?
The ball rolls $r/\sin(30)$ distance, and gravity does $(mg)d\cos(\theta) = mg(r/\sin 30)\sin 30 = mgr$ work.
What if the ball is at the end of a pendulum— attached to a rod of length r? How much work from the time the rod is parallel to the ground to the bottom of its swing?
Use the integral $W = \int F \cdot ds$. Since the ball is swinging along a circle of constant radius, we can write $ds = rd\theta$.

$$\begin{aligned}
W &= \int F \cdot ds \\
&= \int_{\theta=0}^{\pi/2} mg\cos\theta r \cdot d\theta \\
&= mgr\sin\theta|_0^{\pi/2} \\
&= mgr.
\end{aligned}$$

It's the same!

Lines

Consider the line given by $ax + by = c$.

1. Show that the points on the line are the solutions to an equation of the form $u \cdot \langle x, y\rangle = k$.

   $\langle a, b\rangle \cdot \langle x, y\rangle = c$.

2. Show that the points on the line can be expressed in the form $\{(x_0, y_0) + tv : t \in \mathbb{R}\}$ for some (x_0, y_0) and v.

   If $v = (-b, a)$ and $ax_0 + by_0 = c$, then $a(x_0 - tb) + b(y_0 - ta) = ax_0 + by_0 = c$ as well. In the other direction, if $ax + by = c$ then $a(x - x_0) + b(y - y_0) = 0$. This means that $\langle x - x_0, y - y_0\rangle$ is perpendicular to $\langle a, b\rangle$ and so is of the form tv for $v = (-b, a)$.

3. What is the relation between u and v?

   $u \cdot v = 0$.