10.2/10.4: Areas, lengths, and speed

Wednesday, January 27

Warmup

1. $\sin(-3\pi/2) = 1$
2. $\cos(2\pi/3) = -1/2$
3. $\cos(-\theta) = \theta$
4. $\sin(-\theta) = -\sin\theta$
5. $\tan(\theta + \pi) = \tan\theta$
6. $\sin(\theta + \pi) = -\sin\theta$
7. $u = 2x^2$; $du = 4x\,dx$
8. $x = e^{2t}$; $dx = 2e^{2t}\,dt$
9. $y = \sqrt{x}$; $dy = \frac{1}{2\sqrt{x}}\,dx$

1. Two runners start at the same spot. One runs east at 3m/s; the other runs north at 4m/s. What is the distance between the runners as a function of time?

   $3^2 + 4^2 = 5^2$, so $5m/s$.

2. A car drives with velocity $\sin(t)$ m/s for 2π seconds. How far is the car from where it started?

   The car is exactly where it started.

3. How far did the car drive?

   $D = \int_0^{2\pi} |\sin t|\,dt = 2\int_{t=0}^{\pi} \sin t\,dt = 2(\cos 0 - \cos\pi) = 4$. The car drove 4 meters.

Calculus with Parametric Curves

A cannonball is fired from ground level. As a function of the number of seconds t, its x-velocity (in m/s) is given by $dx/dt = 15$ and its y-velocity is given (approximately) by $dy/dt = 20 - 10t$.

1. What is the cannonball's speed as a function of time?

   The speed as a function of time is $\sqrt{15^2 + (20 - 10t)^2} = 5\sqrt{(2t-4)^2 + 9}$.

2. What is the length of the cannonball's arc through the air?

   The cannonball is in the air for 4 seconds, so the length of the arc is

   $$\begin{aligned} s &= \int_{t=0}^{4} ds \\ &= 5\int_{t=0}^{4} \sqrt{(2t-4)^2 + 9}\,dt. \end{aligned}$$

   which can be solved by making the trig substitution $2t - 4 = 3\tan\theta$ and using the identity $\tan^2\theta + 1 = \sec^2\theta$.

3. What is the area under the cannonball's trajectory? (You can find this with or without eliminating the parameter.)

   Integrating the two equations for the velocities gives that $x = 15t$ and $y = 20t - 5t^2$.

   Without eliminating the parameter:

$$
\begin{aligned}
A &= \int_{t=0}^{4} y(dx/dt)dt \\
&= \int_{t=0}^{4} (20t - 5t^2)15\,dt \\
&= 75\int_{t=0}^{4} 4t - t^2\,dt \\
&= 75(2t^2 - t^3/3)|_{t=0}^{4} \\
&= 75(32 - 64/3) \\
&= 800.
\end{aligned}
$$

With eliminating the parameter: get $y = \frac{4}{3}x - \frac{1}{45}x^2$. Also, the cannonball travels 60 meters in the 4 seconds it is airborne, so

$$
\begin{aligned}
A &= \int_{x=0}^{60} \frac{4}{3}x - 45x^2\,dx \\
&= \frac{2}{3}x^2 - \frac{1}{135}x^3|_{x=0}^{60} \\
&= 800.
\end{aligned}
$$

Polar Coordinates

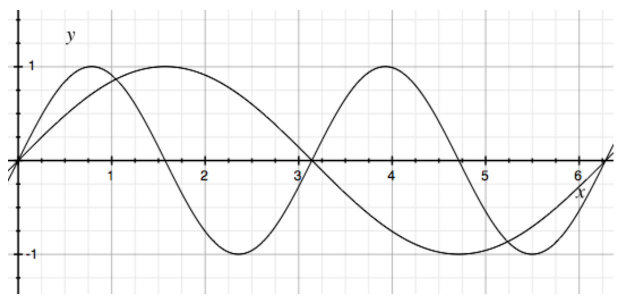
1. Area of a circle with radius r is: πr^2
2. Area of a sector of a circle with radius r, angle θ is: $\pi r^2(\theta/2\pi) = \frac{1}{2}r^2\theta$
3. Find all points of interesection of the given curves: $r = \sin\theta, r = \sin 2\theta$.

   There are two ways for the curves to intersect: either the radii are equal when the angles are pointing in the same direction, or one radius is the negative of the other when the angles are pointed in the opposite direction. In other words, one of the two must hold:

$$\sin 2\theta = \sin\theta$$
$$\sin 2(\theta+\pi) = -\sin\theta$$

   Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$ (and $\sin 2(\theta+\pi) = \sin 2\theta$) and grouping terms gives the equations $\sin\theta(2\cos\theta+1) = 0$ and $\sin\theta(2\cos\theta-1) = 0$, giving the angles $\theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$ ($\theta = 2\pi$ is redundant with $\theta = 0$). Interestingly enough, these correspond to only 3 points in the plane: the origin and the angles $\pi/3, 2\pi/3$. This is because when $\sin\theta$ is negative, the value of r for the first curve is also negative, meaning that the particle traces a single circle twice.

   If you don't have the trig identity at hand, try to sketch the functions and make a guess at what the angles are. Here is a plot of $y = \sin x$ and $y = \sin 2x$, for good measure. The curves intersect at 0, π and 2π, and it looks like $\pm\pi/3$ is a good guess for the other two points of intersection.

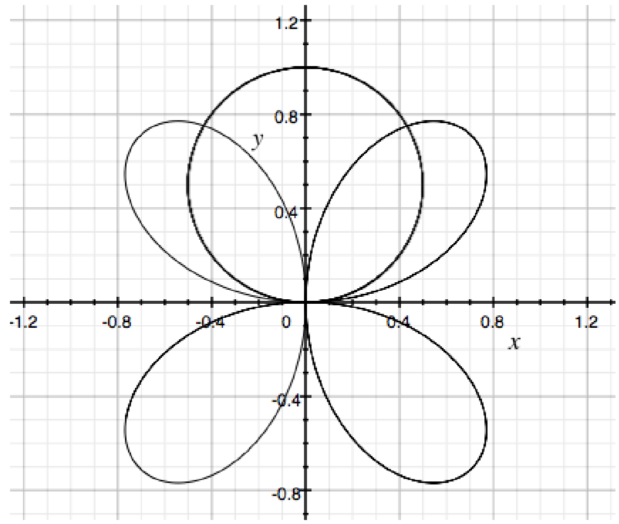


4. Find the area that lies inside both curves: $r = \sin\theta$, $r = \sin 2\theta$.

   Just look at the intersection in the first quadrant: as mentioned previously, the point of intersection is $\theta = \pi/3$. So split the integral into two parts, and use the identity $\cos 2\theta = 1 - 2\sin^2\theta$:

$$\begin{aligned}
A &= \int_{\theta=0}^{\pi/3} \frac{1}{2}\sin^2\theta\, d\theta + \int_{\theta=\pi/3}^{\pi/2} \frac{1}{2}\sin^2 2\theta\, d\theta \\
&= \frac{1}{4}\int_{\theta=0}^{\pi/3} (1-\cos 2\theta)\, d\theta + \frac{1}{4}\int_{\theta=\pi/3}^{\pi/2} (1-\cos 4\theta)\, d\theta \\
&= \frac{\pi}{8} - \frac{1}{4}\left(\int_0^{\pi/3} \cos 2\theta\, d\theta + \int_{\pi/3}^{\pi/2} \cos 4\theta\, d\theta\right) \\
&= \frac{\pi}{8} - \frac{1}{4}\left(\frac{1}{2}\sin 2\theta|_0^{\pi/3} + \frac{1}{4}\sin 4\theta|_{\pi/3}^{\pi/2}\right) \\
&= \frac{\pi}{8} - \frac{1}{4}\left(\sqrt{3}/4 + \sqrt{3}/8\right) \\
&= \frac{\pi}{8} - \frac{3\sqrt{3}}{32}.
\end{aligned}$$

The total area between the two curves is therefore twice this amount: $\frac{\pi}{4} - \frac{3\sqrt{3}}{16}$.



5. Which has a greater area: the four-petaled rose $r = \sin 2\theta$ or the eight-petaled rose $r = \sin 4\theta$?

   It's the same for both! Look at the area traced out for an arbitrary $r = \sin k\theta$ (with even k) for $0 \leq \theta \leq 2\pi$:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} \sin^2 k\theta \, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (1 - \cos 2k\theta)/2 \, d\theta \\ &= \frac{1}{4}\int_0^{2\pi} (1 - \cos 2k\theta) \, d\theta \\ &= \pi/2, \end{aligned}$$

   since the integral of the cosine function from 0 to 2π is zero. (This doesn't happen for odd multiples of θ because then the loop traces over itself twice, and so covers only half the area.)

Bonus

Draw pictures explaining the formulas for (1) arc lenth in parametric coordinates, (2) surface area in parametric coordinates, (3) Area enclosed by polar coordinates.