16.5-7: Surface Integrals

Wednesday, April 27

Vector Field Identities

Prove:

1. curl(curl $\mathbf{F}$) = grad(div($\mathbf{F}$)) - $\nabla^2\mathbf{F}$

$$\begin{aligned}
curl(curl(\mathbf{F})) &= curl(\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle) \\
&= \langle (Q_x - P_y)_y - (P_z - R_x)_z, (R_y - Q_z)_z - (Q_x - P_y)_x, (P_z - R_x)_x - (R_y - Q_z)_y\rangle \\
&= \langle Q_{xy} - P_{yy} - P_{zz} + R_{xz}, R_{yz} - Q_{zz} - Q_{xx} + P_{yx}, P_{zx} - R_{xx} - R_{yy} + Q_{zy}\rangle \\
&= \langle P_{xx} + Q_{xy} + R_{xz}, P_{yx} + Q_{yy} + R_{yz}, P_{zx} + Q_{zy} + R_{zz}\rangle - \nabla^2\mathbf{F} \\
&= \nabla(P_x + Q_y + R_z) - \nabla^2\mathbf{F} \\
&= \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}.
\end{aligned}$$

2. curl($f\mathbf{F}$) = f curl($\mathbf{F}$) + $(\nabla f)\times\mathbf{F}$

$$\begin{aligned}
\nabla\times(f\mathbf{F}) &= \langle (fR)_y - (fQ)_z, (fP)_z - (fR)_x, (fQ)_x - (fP)_y\rangle \\
&= \langle f_yR + fR_y - f_zQ - fQ_z, f_zP + fP_z - f_xR - fR_x, f_xQ - fQ_x - f_yP - fP_y\rangle \\
&= f\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle + \langle f_yR - f_zQ, f_zP - f_xR, f_xQ - f_yP\rangle \\
&= f(\nabla\times\mathbf{F}) + (\nabla f)\times\mathbf{F}.
\end{aligned}$$

Surface Area

$$A(S) = \iint_{(u,v)\in D} |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Find a parametric representation for the part of the sphere $x^2+y^2+z^2$ that lies above the cone $z = \sqrt{x^2+y^2}$.
Bonus: find its area.
In spherical coordinates: the domain $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/4$.

Surface Integrals

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_r \times \mathbf{r}_v|\,dA$$

$$\iint_S f(x,y,z)\,dA = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$$

Find $\iint_S x\,dS$ where S is the triangular region with vertices $(1,0,0), (0,-2,0), (0,0,4)$.

(16.7.11) The vertices satisfy the equation $4x - 2y + z = 4$, so go with the function $z = 4 - 4x + 2y$. Then

$$\begin{aligned}
\iint_S x\,dS &= \iint_{x,y} x\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA \\
&= \int_{x=0}^{1}\int_{y=2x-2}^{0} x\sqrt{21}\,dy\,dx \\
&= \sqrt{21}\int_{x=0}^{1} x(2-2x)\,dx \\
&= \sqrt{21}\int_{x=0}^{1} 2x - 2x^2\,dx \\
&= \sqrt{21}[x^2 - 2x^3/3]_0^1 \\
&= \sqrt{21}/3.
\end{aligned}$$