16.3-4: Fundamental Theorem, Green's Theorem

Wednesday, April 20

Conservative Functions

Decide whether $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

1. $\mathbf{F}(x, y) = (xy + y^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$
2. $\mathbf{F}(x, y) = y^2 e^{xy}\mathbf{i} + (1 + xy)e^{xy}\mathbf{j}$
3. $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$

Find a function f such that $\mathbf{F} = \nabla f$ and evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C:

1. $\mathbf{F}(x, y) = x^2y^3\mathbf{i} + x^3y^2\mathbf{j}, \quad C : \mathbf{r}(t) = \langle t^3 - 2t, t^3 + 2t\rangle, 0 \leq t \leq 1$
2. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$, C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

Theorem 0.1 (Green's Theorem) *Let C be a smooth simple closed curve, positively oriented, enclosing the region D. If P and Q have continuous partial derivatives on an open neighborhood of D then*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = \langle y\cos x - xy\sin x, xy + x\cos x \rangle$ and C is the triangle from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.

True or False?

1. A region D simply connected if any two points in D can be joined by a curve that stays inside D.
2. If $\mathbf{F}$ is conservative on D and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ (where P and Q are continuously differentiable) then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D.
3. If P and Q are continuously differentiable and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D then $\mathbf{F}$ is conservative on D.
4. If there exists a closed curve C in D such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ then $\mathbf{F}$ is conservative on D.
5. If $\mathbf{F}$ is conservative on a region D then there is some function f on D such that $\nabla f = \mathbf{F}$.
6. If two triangles share an edge then the work a force field does on a particle traveling along the two triangles one at a time is the same as if the particle traveled along the quadrilateral boundary of the union of the two triangles.
7. The kinetic energy of an object plus its potential energy due to gravity is always constant.