16.3-4: Fundamental Theorem, Green's Theorem

Wednesday, April 20

Conservative Functions

Decide whether $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

1. $\mathbf{F}(x, y) = (xy + y^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$: $\frac{\partial P}{\partial y} = x + 2y$ and $\frac{\partial Q}{\partial x} = 2x + 2y$, so NO.
2. $\mathbf{F}(x, y) = y^2 e^{xy}\mathbf{i} + (1 + xy)e^{xy}\mathbf{j}$: $\frac{\partial P}{\partial y} = 2ye^{xy} + xy^2 e^{xy}$ and $\frac{\partial Q}{\partial x} = ye^{xy} + xy^2 e^{xy}$, so YES. Both are derivatives of the function $f(x, y) = ye^{xy}$
3. $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x \cos y)\mathbf{j}$: $\frac{\partial P}{\partial y} = e^x + \cos y$ and $\frac{\partial Q}{\partial x} = e^x + \cos y$, so YES. Both are derivatives of the function $f(x, y) = ye^x + x \sin y$.

Find a function f such that $\mathbf{F} = \nabla f$ and evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C:

1. $\mathbf{F}(x, y) = x^2 y^3 \mathbf{i} + x^3 y^2 \mathbf{j}, \quad C : \mathbf{r}(t) = \langle t^3 - 2t, t^3 + 2t \rangle, 0 \le t \le 1.$

   $f(x, y) = \frac{1}{3}x^3 y^3$. The point starts at $(0, 0)$ and ends at $(-1, 3)$, so the potential goes from 0 to -9. -9 units of work are done.
2. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$, C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

   $f(x, y, z) = xyz + z^2$, so the potential goes from 0 to 81. 81 units of work are done.

Theorem 0.1 (Green's Theorem) *Let C be a smooth simple closed curve, positively oriented, enclosing the region D. If P and Q have continuous partial derivatives on an open neighborhood of D then*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle y \cos x - xy \sin x, xy + x \cos x \rangle$ and C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$.
Note that the path is in the mathematically negative direction.

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C P\,dx + Q\,dy \\
&= -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= -\iint_D (y + \cos x - x \sin x) - (\cos x - x \sin x)\, dA \\
&= -\iint_D y\, dA \\
&= -\int_{x=0}^{2} \int_{y=0}^{4-2x} y\, dy\, dx \\
&= -\int_{x=0}^{2} \frac{1}{2}(4-2x)^2\, dx \\
&= -2 \int_{x=0}^{2} 4 - 4x + x^2\, dx \\
&= -2[4x - 2x^2 + \frac{1}{3}x^3]_0^2 \\
&= -16/3.
\end{aligned}
$$

True or False?

1. A region D simply connected if any two points in D can be joined by a curve that stays inside D. FALSE: this only describes a connected region (mostly). D must also not have any "holes," so a doughnut shape would be a counterexample.

2. If $\mathbf{F}$ is conservative on D and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ (where P and Q are continuously differentiable) then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D. TRUE.

3. If P and Q are continuously differentiable and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D then $\mathbf{F}$ is conservative on D. FALSE: the domain must be simply connected.

4. If there exists a closed curve C in D such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ then $\mathbf{F}$ is conservative on D. FALSE: the condition is that this holds for *all* curves in D.

5. If $\mathbf{F}$ is conservative on a region D then there is some function f on D such that $\nabla f = \mathbf{F}$. TRUE.

6. If two triangles share an edge then the work a force field does on a particle traveling along the two triangles one at a time is the same as if the particle traveled along the quadrilateral boundary of the union of the two triangles. FALSE: only if the particle travels along both triangles in the same direction, in which case it's true (the integrals over the shared edge have to cancel out).

7. The kinetic energy of an object plus its potential energy due to gravity is always constant: depends on whether you assume that gravity is the only thing doing work on an object. If yes, then true, but I would say in general FALSE.