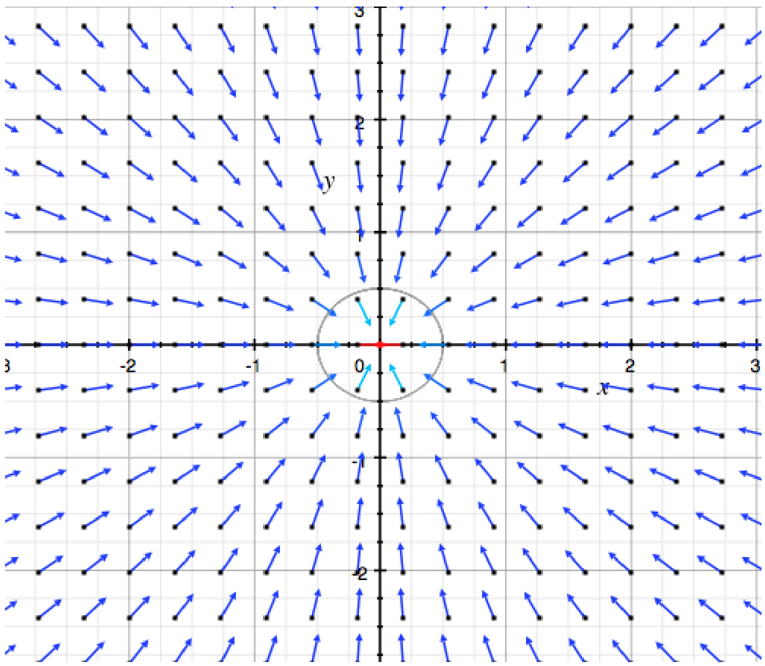
16.1-2: Vector Fields, Line Integrals

Wednesday, April 13

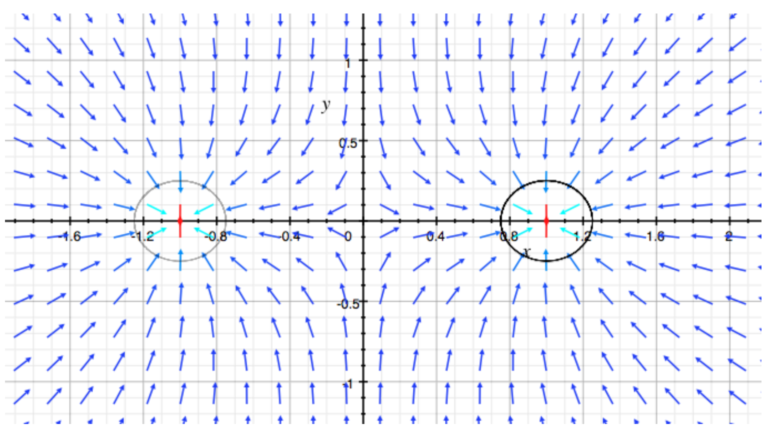
Vector Fields

Sketch the force fields generated by...

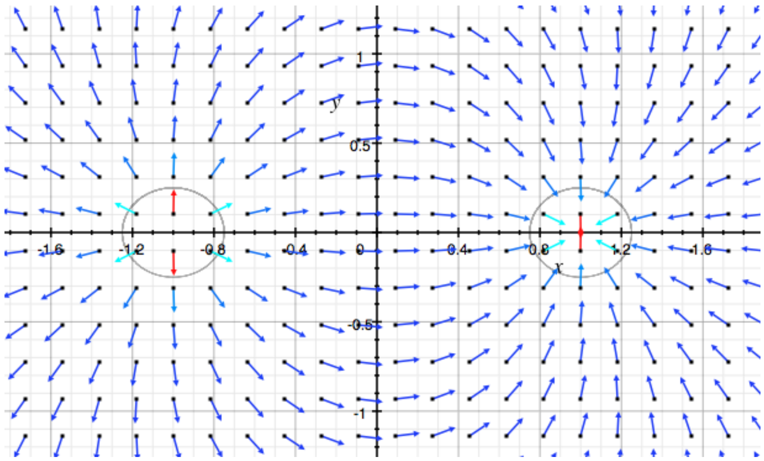
1. A planet



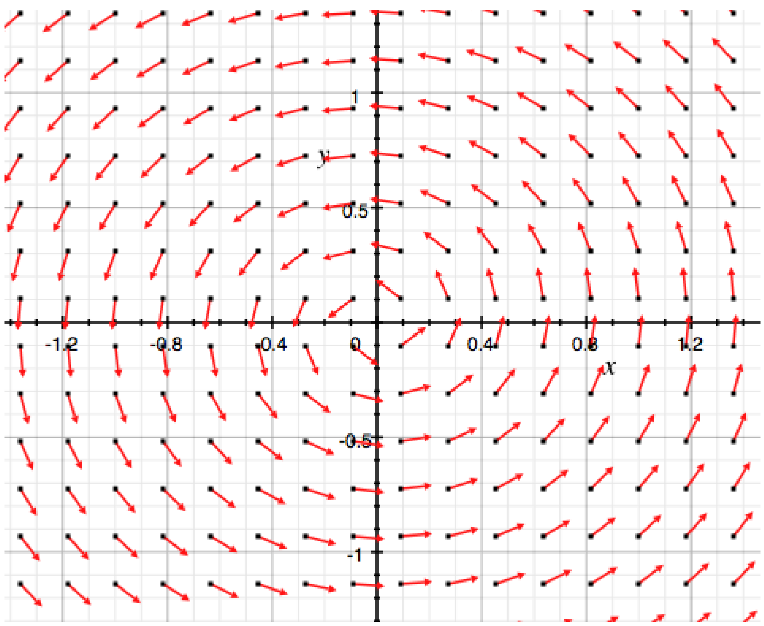
2. Two planets with the same mass



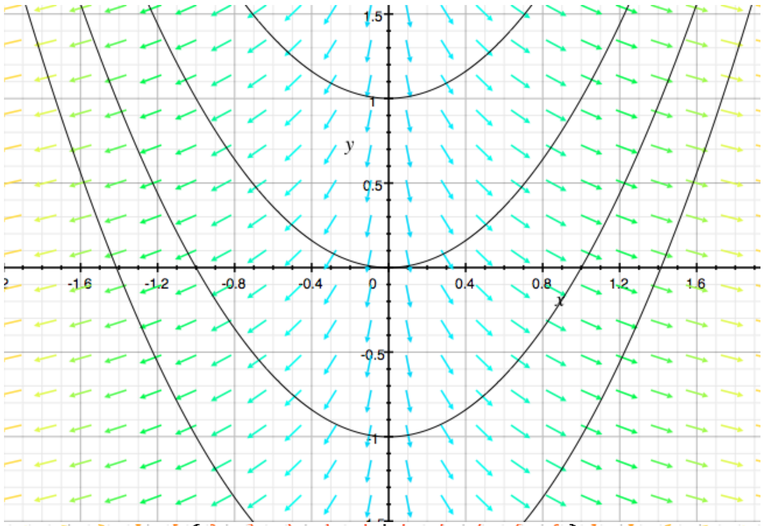
3. Two particles with charges Q and $-Q$

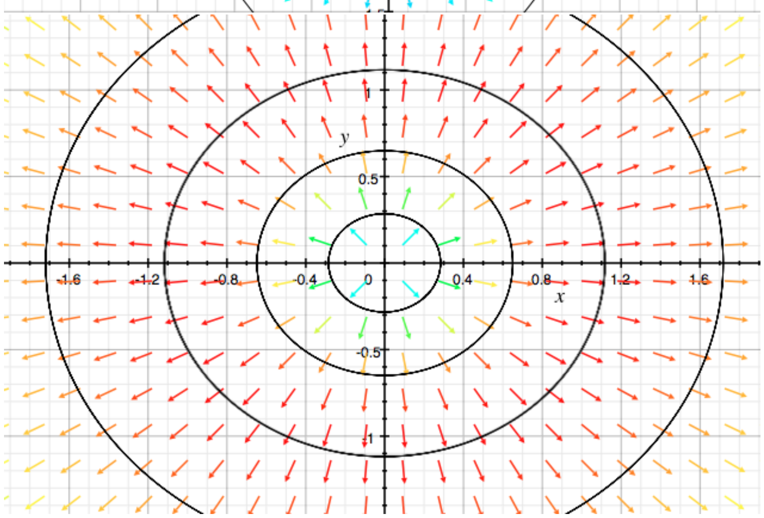


4. The function $\mathbf{F}(x,y) = \dfrac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2+y^2}}$



Sketch the level curves and gradient vector field of $f(x,y) = x^2 - y$ and $g(x,y) = \ln(1 + x^2 + y^2)$.





Line Integrals

Find the line integral $\int_C xy^4\,ds$ where C is the right half of the circle $x^2 + y^2 = 16$. Parametrize the curve with $x = 4\cos t, y = 4\sin t$ as t goes from $-\pi/2$ to $\pi/2$. Then

$$\begin{aligned}
\int_C xy^4\,ds &= \int_{t=-\pi/2}^{\pi/2} 4^5 \cos t \sin^4 t \sqrt{x'(t)^2 + y'(t)^2}\,dt \\
&= \int_{t=-\pi/2}^{\pi/2} 4^5 \cos t \sin^4 t \cdot (4)\,dt \\
&= 4^6 [\frac{1}{5} \sin^5 t]_{-\pi/2}^{\pi/2} \\
&= 2 \cdot 4^6.
\end{aligned}$$

Find the line integral $\int_C (x+2y)\,dx + x^2\,dy$, where C consists of line segments from $(0,0)$ to $(2,1)$ and from $(2,1)$ to $(3,0)$.
Write this as the sum of two separate integrals, one for each line segment. Parametrize the first as $x = 2t, y = t$ and the second as $x = t, y = 3-t$. Then the integral becomes

$$\begin{aligned}
\int_C (x+2y)\,dx + x^2\,dy &= \int_{t=0}^{1} (x+2y)x'(t)\,dt + x^2 y'(t)\,dt + \int_{t=2}^{3} (x+2y)x'(t)\,dt + x^2 y'(t)\,dt \\
&= \int_{t=0}^{1} 4t \cdot 2 \cdot dt + 4t^2\,dt + \int_{t=2}^{3} (6-t)\,dt + t^2(-1)\,dt \\
&= [4t^2 + \frac{4}{3}t^3]_0^1 + [6t - t^2/2 - t^3/3]_2^3 \\
&= (4 + 4/3) + (18 - 9/2 - 9 - 12 + 2 + 8/3) \\
&= 5/2.
\end{aligned}$$