15.2-3: Double Integrals, Polar Coordinates

Wednesday, March 30

Rectangular

Find the volume of the solid with the domain $\{(x,y,z) : x^2 + y^2 \leq 1, y \geq z, x \geq 0, z \geq 0\}$.

Integrate over the xy-plane with $f(x,y) = y$ (taking care of the bounds $z \geq 0$ and $y \geq z$). The relevant domain is a quarter circle with $x, y \geq 0$ and $x^2 + y^2 \leq 1$, so get the integral

$$\begin{aligned}
\iint_D f(x,y)\,dA &= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y\,dy\,dx \\
&= \int_{x=0}^{1} \frac{1}{2}(1-x^2)\,dx \\
&= \frac{1}{2}(x - x^3/3)|0^1 \\
&= \frac{1}{3}.
\end{aligned}$$

Polar

Do the previous problem again, but this time with polar coordinates!

As mentioned before, the relevant domain is a quarter circle, so make the substitution $y = r\sin\theta$ and get

$$\begin{aligned}
\iint_D f(r,\theta)\,dA &= \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \sin\theta\,dr\,d\theta \\
&= \frac{1}{3}\int_{\theta=0}^{\pi/2} \sin\theta\,d\theta \\
&= \frac{1}{3}(-\cos\theta)|_0^{\pi/2} \\
&= \frac{1}{3}.
\end{aligned}$$

Oh good, it's the same answer! Also, since this is a "rectangle" in polar coordinates you can swap the order of integration and it won't be any easier or harder.
Better yet: if you're integrating over a rectangle and you can decompose your function as $f(x,y) = g(x)h(y)$, then in general you will get the relation $\iint f(x,y)\,dA = \left(\int g(x)\,dx\right)\left(\int h(y)\,dy\right)$.

Why is the variable substitution $dA = r\,dr\,d\theta$ correct? Draw a picture.
You should get a section of a circle which for small $dr, d\theta$ can be approximated by a rectangle with side lengths dr and $rd\theta$. A picture is in the book, section 15.3.

Draw some domains that are well-suited for the following coordinate systems:

- Cartesian but not polar: rectangles, triangles, parabolic or polynomial boundaries.
- polar but not Cartesian: triangular wedges of a circle, partial tubes.
- both: circles, semicircles are both okay.
- neither: most shapes, to be honest.

Find the volume of a sphere with radius a.

Go with polar coordinates here, remembering to count both top and bottom hemispheres (so $f(r\theta) = 2\sqrt{a^2 - r^2}$):

$$\begin{aligned}\int_{r=0}^{a} \int_{\theta=0}^{2\pi} 2\sqrt{a^2 - r^2} r \, dr \, d\theta &= 4\pi \int_{r=0}^{a} r\sqrt{a^2 - r^2} \, dr \\ &= 4\pi \frac{-1}{3} (a^2 - r^2)^{3/2} |0^a \\ &= \frac{4}{3}\pi a^3.\end{aligned}$$