15.2-3: Double Integrals, Polar Coordinates
Wednesday, March 30

Rectangular

Find the volume of the solid with the domain \{(x, y, z) : x^2 + y^2 \leq 1, y \geq z, x \geq 0, z \geq 0\}.

Integrate over the xy-plane with \(f(x, y) = y\) (taking care of the bounds \(z \geq 0\) and \(y \geq z\)). The relevant domain is a quarter circle with \(x, y \geq 0\) and \(x^2 + y^2 \leq 1\), so get the integral

\[
\iiint_{D} f(x, y) \, dA = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y \, dy \, dx
\]

\[
= \int_{x=0}^{1} \frac{1}{2} (1 - x^2) \, dx
\]

\[
= \frac{1}{2} (x - x^3/3) \bigg|_{0}^{1}
\]

\[
= \frac{1}{3}.
\]

Polar

Do the previous problem again, but this time with polar coordinates!

As mentioned before, the relevant domain is a quarter circle, so make the substitution \(y = r \sin \theta\) and get

\[
\iiint_{D} f(r, \theta) \, dA = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \sin \theta \, dr \, d\theta
\]

\[
= \frac{1}{3} \int_{\theta=0}^{\pi/2} \sin \theta \, d\theta
\]

\[
= \frac{1}{3} (-\cos \theta) \bigg|_{0}^{\pi/2}
\]

\[
= \frac{1}{3}.
\]

Oh good, it’s the same answer! Also, since this is a “rectangle” in polar coordinates you can swap the order of integration and it won’t be any easier or harder.

Better yet: if you’re integrating over a rectangle and you can decompose your function as \(f(x, y) = g(x)h(y)\), then in general you will get the relation \(\iiint f(x, y) \, dA = \left(\int g(x) \, dx\right) \left(\int h(y) \, dy\right)\).

Why is the variable substitution \(dA = r \, dr \, d\theta\) correct? Draw a picture.
You should get a section of a circle which for small \(dr, d\theta\) can be approximated by a rectangle with side lengths \(dr\) and \(r \, d\theta\). A picture is in the book, section 15.3.
Draw some domains that are well-suited for the following coordinate systems:

- Cartesian but not polar: rectangles, triangles, parabolic or polynomial boundaries.
- polar but not Cartesian: triangular wedges of a circle, partial tubes.
- both: circles, semicircles are both okay.
- neither: most shapes, to be honest.

Find the volume of a sphere with radius \( a \).

Go with polar coordinates here, remembering to count both top and bottom hemispheres (so \( f(r\theta) = 2\sqrt{a^2 - r^2} \)):

\[
\int_{r=0}^{a} \int_{\theta=0}^{2\pi} 2\sqrt{a^2 - r^2} r \, dr \, d\theta = 4\pi \int_{r=0}^{a} r \sqrt{a^2 - r^2} \, dr \\
= 4\pi \left( \frac{1}{3} (a^2 - r^2)^{3/2} \right) \bigg|_0^a \\
= \frac{4}{3} \pi a^3.
\]