10.1/10.5: Parametric Curves and Conic Sections

Wednesday, January 20

Welcome!

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Syllabus Recap

Homework will be assigned on Wednesdays and a quiz closely related to the homework will be given on the following Wednesday. Assignments will not be collected or graded, but solutions will normally be posted on the professor's webpage the Tuesday before the quiz and taken down about a week later.
Quizzes will normally be given on Wednesdays and will be based on the previous week's homework. Makeups are not permitted, but only your top 10 scores (out of about 12) will count toward your final grade.
For all administrative questions, consult Professor Stankova or Thomas Brown. See the course syllabus for details.

Warmup

Evaluate:

1. $(a+b)(a-b) = a^2 - b^2$
2. $\cos^2(x) + \sin^2(x) = 1$
3. $\tan^2(x) + 1 = \sec^2(x)$
4. $e^x \cdot e^{-x} = 1$
5. $\sec^2(x) - \tan^2(x) = 1$
6. $\dfrac{1}{3^{-2}} = 9$
7. $1 - \sin^2(x/2) = \cos^2(x)$
8. $(2\cos 2x)^2 + (2\sin 2x)^2 = 4$
9. $\sqrt{e^{4x}} = e^{2x}$

Make a rough sketch:

1. $(x-1)^2 + (y+2)^2 = 4$: Circle of radius 2 and center $(1, -2)$
2. $x = 3(y-2)^2 + 1$: Right-opening parabola with base at $(2, 1)$
3. $(x/3)^2 + (2(y-1))^2 = 1$: Short, fat ellipse with center at $(0, 1)$

Does the function have an inverse?

1. $f(t) = 3t + 5$: $f^{-1}(t) = (t-5)/3$
2. $f(t) = t^2 - 2$: NO
3. $f(t) = e^t$: $f^{-1}(t) = \ln t$
4. $f(t) = t + 3\sin 2t$: NO
5. $f(t) = \sqrt{t}$: $f^{-1}(t) = t^2$
6. $f(t) = t^3 - 1$: $f^{-1}(t) = \sqrt[3]{t+1}$

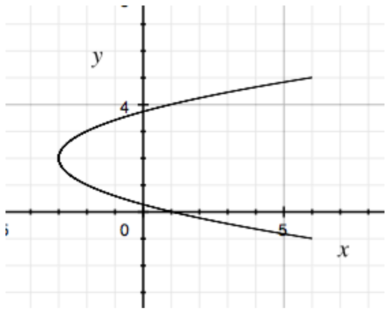
Parametric Curves

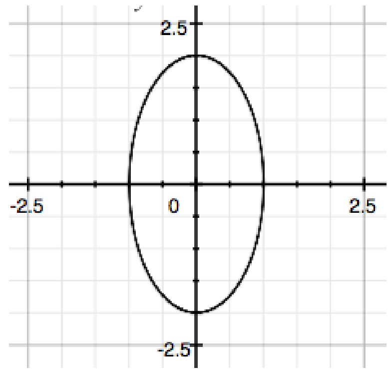
Eliminate the parameter to find a Cartesian equation of the curve. Sketch the curve and indicate with an arrow the direction in which the curve is traced.

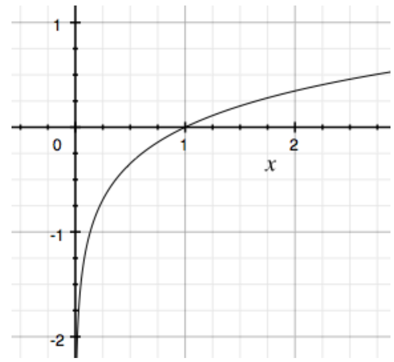
1. $x = t^2 - 3, y = t + 2, -3 \le t \le 3$: The second equation is invertible, so say $t = y - 2$ and get $x = (y-2)^2 - 3$, a parabola. The particle starts at the bottom and travels upwards.

2. $x = \sin(2t), y = -2\cos(2t), 0 \le t \le \pi$: $y/2 = \cos(2t)$, so $x^2 + (y/2)^2 = 1$: an ellipse. The particle starts at the bottom $(0, -2)$ and makes one full circle counterclockwise.

3. $x = t^2, y = \ln t, 0 < t \le \infty$: Since $t > 0$ the first equation is invertible. So $t = \sqrt{x}$ and $y = \ln\sqrt{x} = \frac{1}{2}\ln x$. OR: $t = e^y$ and $x = (e^y)^2 = e^{2y}$. As $t \to \infty$ the particle moves upward and to the right.







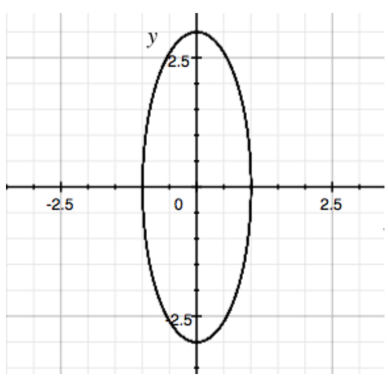
Determine what type of shape each equation represents, then sketch. If the shape is an ellipse or a hyperbola, express it in parametric form.

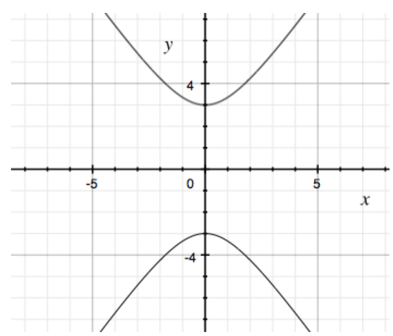
1. $x^2 + (y/3)^2 = 1$: Ellipse. Look to use the identity $\sin^2(t) + \cos^2(t) = 1$ and get $x = \sin t, y/3 = \cos t$. Other representations (such as $x = -\cos 2t, y/3 = \sin 2t$) would also trace out an ellipse, but the particle's speed and direction will vary depending on your choice.

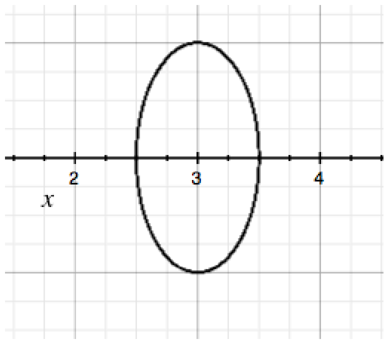
2. $x^2/4 - y^2/9 = -1$: Hyperbola. Use $\cosh^2 t - \sinh^2 t = 1$ and get $x/2 = \sinh t, y/3 = \cosh t$. Since cosh is always positive this will only trace the top half of the hyperbola, so we could describe the bottom half with $x/2 = \sinh t, y/3 = -\cosh t$.

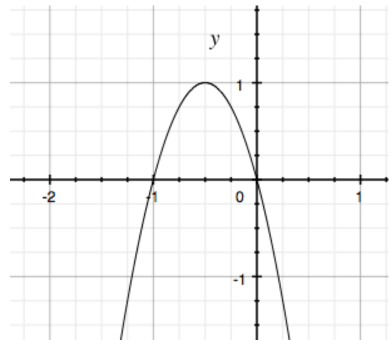
3. $4(x-3)^2 = 1 - y^2$: $4(x-3)^2 + y^2 = 1$, so it's a tall skinny ellipse centered at $(3, 0)$. A valid parametric equation tracing the ellipse would be $2(x-3) = \cos t$ (so $x = 3 + (\cos t)/2$) , $y = \sin t$.

4. $(2x+1)^2 + y = 1$: $y = 1 - (2x+1)^2$, so it's a downward facing parabola with a maximum at $-1/2, 1$.









Bonus

Prove that $\cosh^2(x) - \sinh^2(x) = 1$ for all $x \in \mathbb{R}$.

$$\cosh^2(x) - \sinh^2(x) = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1.$$

Compare the curves represented by the following parametric equations:

1. $x = t^3, y = t^2$
2. $x = t^6, y = t^4$
3. $x = e^{-3t}, y = e^{-2t}$

All three lie on the curve $x^2 = y^3$, but trace out different parts of it. The first particle moves from left to right and traces out the whole curve. The second particle starts at the right, moves to the origin as $t \to 0$, then turns around and moves leftward again (note that $x, y \geq 0$ for all t). The third particle starts at the right and approaches the origin (never reaching it!) as $t \to \infty$.