Review 3: Practice Final

Friday, May 6

1. Find the point(s) on the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$ that maximize the function $f(x, y) = x^2 - 2x + 2y^2$.

   $\nabla f(x, y) = \langle 2x - 2, 4y \rangle$, which is zero only at $(1, 0)$. Since $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$, this is a local minimum rather than a local maximum, so we have to use Lagrange multipliers instead.

   Get the equations

$$\begin{aligned} x^2 + y^2 &= 1 \\ x &= \lambda(x - 1) \\ y &= 2\lambda y. \end{aligned}$$

   The equation $y = 2\lambda y$ has solutions $\lambda = 0, y = 0$ (which gets us the point $(1, 0)$ as before) or $\lambda = 1/2$. This second option implies that $x = -1$ and so $y = 0$. $f(-1, 0) = 3$, which is larger than $f(1, 0)$, so the point $(-1, 0)$ is the unique maximum on the disk.

2. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

$$\begin{aligned} \int_{\rho=0}^{2} \int_{\phi=\pi/4}^{\pi/2} \int_{\theta=0}^{2\pi} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho &= \left(\int_{\rho=0}^{2} \rho^2 \, d\rho \right) \left(\int_{\phi=\pi/4}^{\pi/2} \sin\phi \, d\phi \right) \left(\int_{\theta=0}^{2\pi} d\theta \right) \\ &= (8/3) \cdot (2\pi) \cdot [\cos \pi/4 - \cos \pi/2] \\ &= (8/3) \cdot (2\pi) \cdot \sqrt{2}/2 \\ &= 8\pi\sqrt{2}/3. \end{aligned}$$

3. If $\mathbf{F}(x, y) = \langle 2xe^{-y}, 2y - x^2 e^{-y} \rangle$, evaluate the integral $\int \mathbf{F} \cdot d\mathbf{r}$ where C is a series of line segments from $(1, 0)$ to $(3, 3)$ to $(-4, 7)$ to $(\pi, 3\sqrt{2^\pi})$ to $(2, 1)$.

   If $\mathbf{F} = \langle P, Q \rangle$, observe that $P_y = -2xe^{-y} = Q_x$, so since $\mathbf{F}$ is defined on all of $\mathbb{R}^2$ we can conclude that it is conservative. Inspection (or integrating P with respect to x and Q with respect to y and equating the two) give that $\mathbf{F}$ is the gradient of $f(x, y) = y^2 + x^2 e^{-y}$.

   We may therefore invoke the Fundamental Theorem of Line Integrals, and conclude that $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 1) - f(1, 0) = (1 + 4/e) - (1) = 4/e$.

4. If $\mathbf{F}(x, y) = \langle x, 2y, 3z \rangle$, evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the cube with vertices $(\pm 1, \pm 1, \pm 1)$ (oriented outward).

   [NOTE: The original posted solution was incorrect.]

Consider the faces of the cube where $x = \pm 1$. The normal vectors are $\langle 1, 0, 0\rangle$ when $x = 1$ and $\langle -1, 0, 0\rangle$ where $x = -1$, so $\mathbf{F} \cdot d\mathbf{S} = x = 1$ on the first and $\mathbf{F} \cdot d\mathbf{S} = -x = 1$ on the second. Since these are constants, on each face we get $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S dS$, which is just the area of the face. Since the faces of the cube all have edge length 4, the surface integrals over the faces $x = 1$ and $x = -1$ are both equal to 4.

Similarly, the surface integrals over the faces $y = \pm 1$ and $z = \pm 1$ are both 8 and 12, respectively, so the whole surface integral is equal to 48.

Alternately: use the divergence theorem [optional: you won't need to know this, but it may be useful!] $\nabla \cdot \mathbf{F} = 6$ uniformly, so $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V 6\, dV$, which is 6 times the volume of the cube, which is 48.

5. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle yz, 2xz, e^{xy} \rangle$ and C is the circle $x^2 + y^2 = 16, z = 5$, oriented counterclockwise as viewed from above.

This calls for Stokes' Theorem. Based on the orientation given, the normal vector to the surface is $\langle 0, 0, 1 \rangle$, so $(\nabla \times \mathbf{F}) \cdot d\mathbf{S} = Q_x - P_y$, the same as for Green's Theorem. Therefore,

$$\begin{aligned}
\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_S Q_x - P_y \, dS \\
&= \int_S 2z - z \, dS \\
&= 5 \int_S dS \\
&= 5 \cdot 16\pi \\
&= 80\pi.
\end{aligned}$$