14.3-6: Recap

Monday, March 7

Hill climbing

A hiker is climbing a hill. The height of the hill is given by $H(x, y) = \max(-x^2 - 2x - 9y^2 + 36y - 1, 0)$ (so that the height never goes below zero).

1. Sketch a contour map of the hill. In particular, find the curve describing where the hill hits ground level.

   The surface, when above ground level, is an elliptic paraboloid with $H(x, y) = 36 - (x+1)^2 - 9(y-2)^2$. The curve where the hill hits ground level is an ellipse of the form $(x+1)^2 + 9(y-2)^2 = 36$ (semiaxes of lengths 6 and 2, repsectively).

2. Is the height of the hill continuous as a function of x and y? Is it differentiable? If not, find a point and show that it is not continuous/differentiable at that point.

   The height is continuous but not differentiable wherever the hill meets the ground. At the point $(5, 2)$, for example, the directional derivative in the direction $\langle 1, 0\rangle$ is zero but the directional derivative in the direction $\langle -1, 0\rangle$ is 12.

3. The hiker is currently at the location $(2, 1)$. Find H_x and H_y at this point.

   $\langle H_x, H_y\rangle = \langle -2(x+1), -18(y-2)\rangle = \langle -6, 18\rangle$ at the point $(2, 1)$.

4. If the hiker moves in the direction $\langle 1, 1\rangle$, will they be moving uphill or downhill? $\langle -4, -1\rangle$? $\langle -1, 3\rangle$?

   Whether $\langle H_x, H_y\rangle \cdot v$ is positive or negative will tell whether the hiker is moving uphill or downhill. $\langle 1, 1\rangle$ points uphill, $\langle -4, -1\rangle$ points downhill, and $\langle -1, 3\rangle$ points uphill.

5. Find the curve consisting of all points on the hill with height equal to the hiker's current height. Find the slope tangent to the curve at the hiker's current location.

   The current height is 18, so the curve is an ellipse of the form $(x+1)^2 + 9(y-2)^2 = 18$. The Chain Rule gives $dy/dx = -\frac{F_x}{F_y} = -\frac{-2(x+1)}{-18(y-2)} = \frac{1}{3}$.

6. Let $L(x, y)$ be the linear approximation to H at $(2, 1)$. Plot the level curve of L that passes through $(2, 1)$. What is its relation to the level curve of H?

   $L(x, y) = 18 + 6(x-2) - 18(y-1)$. If $18 = 18 + 6(x-2) - 18(y-1)$ (the equation describing the level curve), then $y - 1 = (x-2)/3$. This level curve has the slope given in the previous answer and is tangent to the level curve of H.

7. What direction should the hiker walk in order to (at that instant) be moving neither uphill nor downhill?

   The hiker should walk in the direction $\pm\langle 1, 3\rangle$ (since this keeps the hiker on the level curve from the answer above).

8. Suppose the hiker's location as a function of time is $\langle -t^2 + t + 2, t\rangle$, so that the hiker hits the point $(2, 1)$ at time $t = 1$. Will the hiker be moving uphill or downhill at $t = 1$?

   The hiker's velocity at time $t = 1$ is $\langle -2t + 1, 1\rangle = \langle -1, 1\rangle$, so (since the directional derivative in this direction is positive) the hiker is moving uphill.

   Also: could use Chain rule, or eliminate x and y to find $H(t) = H(x(t), y(t))$ and solve from there.

9. At what point in time (approximately) is the hiker's elevation at a maximum?

   The simplest thing to do is compute $H(x(t), y(t)) = H(-t^2+t+2, t) = 36-(-t^2+t+3)^2-9(t-2)^2 = -t^4 + 2t^3 - 4t^2 + 30t - 9$ (unless this is below zero, in which case the hiker is at ground level). The derivative is $-4t^3 + 6t^2 - 8t + 30$, which is zero at approximately $t = 2.17$.