12.3-12.5: Recap

Monday, February 15

Warmup

Let $\mathbf{u} = \langle 1,2,3\rangle$, $\mathbf{v} = \langle -5,1,1\rangle$, $\mathbf{w} = \langle -3,5,7\rangle$. Find:

1. $\mathbf{u} \cdot \mathbf{v} = 0$
2. $\mathbf{u} \cdot \mathbf{u} = 1^2 + 2^2 + 3^2 = 14$
3. $\mathbf{v} \times \mathbf{w} = -3 + 10 + 21 = 28$
4. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$
5. What does the previous answer tell you about $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$? The three vectors and the zero vector are coplanar. This also means that $\mathbf{w}$ can be written as a linear combiation of $\mathbf{u}$ and $\mathbf{v}$.
6. Make a sketch of the plane $x + y + z = 1$ in the region $x, y, z \geq 0$. *It should look like a triangle with vertices at* $\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle$.
7. What is the relation between the sets described by $\mathbf{u} \cdot x = 1$ and $\mathbf{u} \cdot x = 2$? *They're parallel planes.*

True or False

1. For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

   Visual intuition: up to sign, yes, because (with the zero vector) the triple product describes the volume of a parallelepiped.
2. For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

   Nope. $i \times (i \times j) = i \times k = -j$, but $(i \times i) \times j = 0 \times j = 0$.
3. If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

   Nope. They can be perpendicular.
4. If $\mathbf{u} \times \mathbf{v} = 0$ then $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

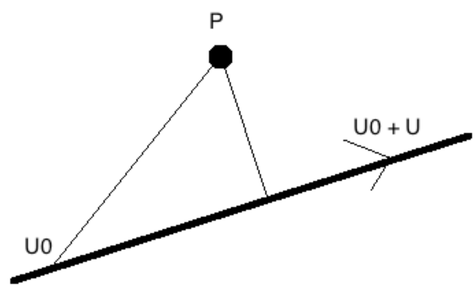
   Nope. They can be parallel.
5. The intersection of two non-parallel planes is always a line.

   Yup.

Three Dimensions

- Find a formula for the distance from a point P_0 to a line of the form $u_0 + tu$. Make a picture first.

  It's the height of a triangle, so $|(P_0 - u_0) \times u|/|u|$, using the properties of the cross product.



- Find the set of points equidistant from two parallel lines of the form $u_0 + t\mathbf{u}$ and $u_1 + t\mathbf{u}$. Make a sketch first and guess what the answer should be before doing any computations.

  The simplest way is to make the right observation: it's a plane lying halfway between the lines, and the normal vector to the plane is a vector describing the shortest path between the two planes.

  However, there is no guarantee that the distance from u_0 to u_1 is also the distance betwen the lines, so we have to make a projection of some sort. The projection vector from u_0 to the line through u_1 is $u' := u\frac{(u_0 - u_1)\cdot u}{|u|^2}$, so define $u_2 = u_1 + u'$.

  This means that $u_0 - u_2$ is the normal vector to the plane, so call it $\mathbf{v}$. The plane, being halfway between the two lines, is given by $\mathbf{v} \cdot x = (\mathbf{v} \cdot u_0 + \mathbf{v} \cdot u_1)/2$.

- Bonus: What if the lines intersect? What if they are skew lines?

  Have fun!

- Given two intersecting planes described by the equations $\mathbf{u} \cdot x = k_1$ and $\mathbf{v} \cdot x = k_2$, find a way to describe the intersection.

  The intersection is a line, and since it lies in both planes it is perpendicular to both normal vectors. It can therefore be written in the form $P + t(\mathbf{u} \times \mathbf{v})$. Then we need to find a suitable P: we have the equations $k_1 = \mathbf{u} \cdot (P + t(\mathbf{u} \times \mathbf{v})) = \mathbf{u} \cdot P$ and similarly $\mathbf{v} \cdot P = k_2$. Trial and error should work from there.

  Note that these equations mean that if $\mathbf{u} = \mathbf{v}$ but $k_1 \neq k_2$ then the planes are parallel and there is no intersection.