10.1,10.2,10.3,10.5: Recap

Monday, January 25

♣ Evaluate:

1. $\sin(5\pi/6) = 1/2$
2. $\cos(5\pi/4) = -\sqrt{2}/2$
3. $\cos(2\pi/3) = -1/2$
4. $\sin(-12\pi) = 0$
5. $\cos(-\pi/6) = -\sqrt{3}/2$
6. $\tan(\pi/4) = 1$
7. $\frac{d}{d\theta}\sin\theta/\theta = \cos\theta/\theta - \sin\theta/\theta^2$
8. $\frac{d}{dx}2\sin(x^2)\cos(x^2) = \frac{d}{dx}\sin(2x^2) = 2x\cos(x^2)$
9. $\frac{d}{dt}e^t(\sin^2 t + \cos^2 t) = \frac{d}{dt}e^t = e^t$

Find Cartesian and parametric equations that describe each of the following:

1. An ellipse centered at $(3, -1)$ with semiaxes of length 2 and 5: $((x-3)/2)^2 + ((y+1)/5)^2 = 1$, or $x = 3 + 2\cos t, y = -1 + 5\sin t$.
2. A parabola opening upward that hits its minimum at $(2, 0)$ $y = k(x-2)^2$, for any $k > 0$, or $x = t + 2, y = t^2$.
3. A hyperbola opening horizontally. $x^2 - y^2 = 1$, or $x = \cosh t, y = \sinh t$.

♡ Describe the path the particle takes, and sketch. Find all points (in time and space) where the line tangent to the curve has slope 1.

1. $x = -3\cos t,\ y = 2\sin t,\ \pi/2 \le t \le 3\pi/2$

   It's a fat ellipse centered at the origin, starting at $(0, 2)$ and moving clockwise to $(0, -2)$.

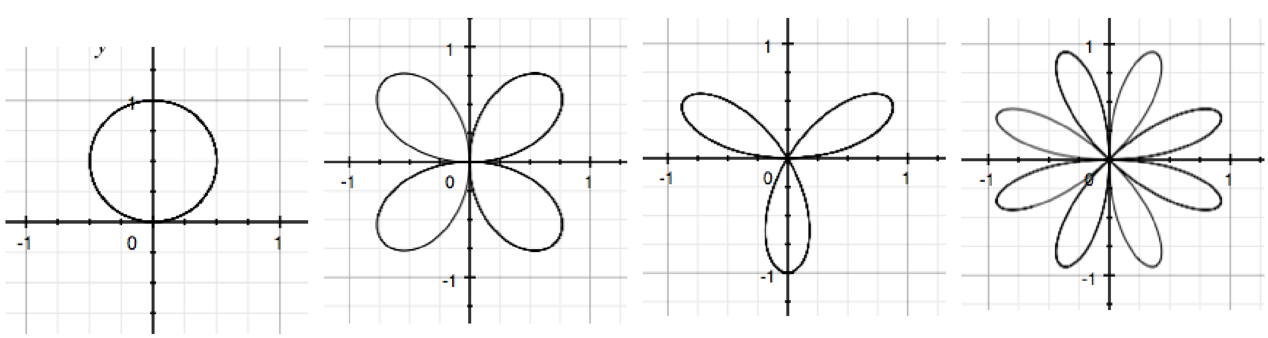
2. $x = \sin 2t,\ y = 1 - \cos^2 2t,\ 0 \le t \le 2\pi$

   Note that $y = \sin^2 2t = x^2$, so the point moves back and forth twice between $(-1, 1)$ and $(1, 1)$ along a parabola.

3. $x = t^2 - 1,\ y = t^2 - 1,\ -\infty < t < \infty$

   $x = y$, so the particle stays on the path $y = x$. It moves down and left until $t = 0$ where it stops at $(-1, -1)$, then reverses course.

♢ Below are graphs of the polar equations $r = \sin\theta$, $r = \sin 2\theta$, $r = \sin 3\theta$, and $r = \sin 4\theta$. Explain. What does the path of the particle look like?



In the odd cases, each loop is visited twice. This is true because for odd numbers k, $\sin(k(\theta+\pi)) = -\sin(k\theta)$, so when the angle is pointing in the opposite direction r has the opposite sign.

For $\sin 2t, \sin 3t$ the loops are visited in clockwise order starting from the first quadrant. For $\sin 4t$ the particle visits every third loop going in clockwise order, starting from the bottom loop in the first quadrant.

What do you think the graphs of $r = \cos\theta$, $r = \cos 2\theta, r = \cos 3\theta, r = \cos 4\theta$ look like?
They are just rotated versions of the previous 4 graphs, since $\cos(\theta) = \sin(\pi/2 - \theta)$.
♠ True or False? Explain your reasoning.

1. The curve defined by any set of parametric equations $(x, y) = (f(t), g(t))$ can also be defined by an equation of the form $y = h(x)$.

   False: $x = \cos\theta, y = \sin\theta$ defines a circle, and y cannot be expressed as a function of x.

2. The curve defined by any equation of the form $y = h(x)$ can also be defined by a set of parametric equations $(x, y) = (f(t), g(t))$.

   True: set $x = t$ and $g(t) = h(x)$.

3. If $dy/dt = 0$ at some point on a curve then the tangent line at that point is horizontal.

   False: if dx/dt is also zero then the tangent line is not necessarily horizontal.

4. If a circle is parametrized as $(x, y) = (\cos t, \sin t)$, then for any t the angle between $(x(t), y(t))$ and the positive x-axis will be equal to t.

   True for circles, but not for ellipses.

5. If $f(\theta) = f(-\theta)$ for all θ, then the curve defined by $r = f(\theta)$ will have a vertical axis of symmetry.

   False: it will have a horizontal axis of symmetry.

6. If $f(\theta) = f(\theta + \pi)$ for all θ, then the curve defined by $r = f(\theta)$ will be unchanged when it is rotated by 180 degrees about the origin.

   True.