16.4-5: Green's Theorem, Curl, Divergence

Monday, April 25

Equations

- $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA$

(16.4.13) Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle y - \cos y, x \sin y\rangle$ and C is the circle $(x-3)^2 + (y+4)^2 = 4$, oriented clockwise.

(16.5.7) Find the curl and divergence of the vector field $\mathbf{F}(x, y, z) = \langle e^x \sin y, e^y \sin z, e^z \sin x\rangle$.

Curl

Consider the vector field $\mathbf{F}(x, y, z) = \langle y, -x, 0 \rangle$.

1. Sketch $\mathbf{F}$ over the xy-plane.
2. What is $\nabla \times \mathbf{F}$? Is $\mathbf{F}$ conservative?
3. Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle in the xy-plane, oriented counterclockwise.
4. Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle in the yz-plane, counterclockwise with respect to the positive x direction.
5. If you place a small paddle wheel at the origin, in what direction should you point its axis to make it spin the fastest?
6. Explain in terms of line integrals how water wheels generate power. Does the concept of the curl of a vector field apply to your explanation?
7. Prove that if a vector field $\mathbf{F}$ is conservative then $\nabla \times \mathbf{F} = 0$.
8. Give a physical explanation of why this should be true. (What is the interpretation of curl involving line integrals?)