16.4-5: Green's Theorem, Curl, Divergence

Monday, April 25

Equations

- $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA$

(16.4.13) Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle y - \cos y, x \sin y\rangle$ and C is the circle $(x - 3)^2 + (y + 4)^2 = 4$, oriented clockwise.
$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sin y - (1 + \sin y) = -1$, so (with a negative sign since the orientation of the curve is clockwise) we get $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = -\iint -1\,dA = A$, the area of the circle, which is 4π.

(16.5.7) Find the curl and divergence of the vector field $\mathbf{F}(x, y, z) = \langle e^x \sin y, e^y \sin z, e^z \sin x\rangle$. [NOTE: the original problem had a different second coordinate; this was a typo.]
Divergence: $\nabla \cdot \mathbf{F} = e^x \sin y + e^y \sin z + e^z \sin x$.
Curl:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^y \sin z & e^z \sin x \end{vmatrix} \\ &= \langle -e^y \cos z, -e^z \cos x, -e^x \cos y\rangle.\end{aligned}$$

Curl

Consider the vector field $\mathbf{F}(x, y, z) = \langle y, -x, 0\rangle$.

1. Sketch $\mathbf{F}$ over the xy-plane.

   The vector at a point is always tangent to the circle (centered at the origin) the point lies on, pointing clockwise, with length proportional to the radius of the circle.

2. What is $\nabla \times \mathbf{F}$? Is $\mathbf{F}$ conservative?

   $\nabla \times \mathbf{F} = \langle 0, 0, -2\rangle$, so the field is not conservative.

3. Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle in the xy-plane, oriented counterclockwise.

   Applying Green's theorem (and using the above answer) gives that the integral is equal to $\iint -2\,dA = -2\pi$, so if an object travels counterclockwise the field does work against it.

4. Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle in the yz-plane, counterclockwise with respect to the positive x direction.

   Apparently we didn't get to the more general Stokes' Theorem in class, so don't worry about this one. But the answer should be zero because the axis of rotation is perpendicular to the direction of the curl vector.

5. If you place a small paddle wheel at the origin, in what direction should you point its axis to make it spin the fastest?

   Point its axis of rotation precisely in the z direction, since this is parallel to the curl.

6. Explain in terms of line integrals how water wheels generate power. Does the concept of the curl of a vector field apply to your explanation?

   As a paddle on the water wheel turns, the river does positive work on it when it is in the water and no work against it when it is not in the water, to the work done over the closed curve is positive.

   The curl is not relevant becuase if our model has non-zero force only in the body of the river then the field is not a continuous function.

7. Prove that if a vector field $\mathbf{F}$ is conservative then $\nabla \times \mathbf{F} = 0$.

   The proof is given for Theorem 3 in the textbook.

8. Give a physical explanation of why this should be true. (What is the interpretation of curl involving line integrals?)

   A physical interpretation of the curl at a point is the infintesimal work done for a line integral in an infintesimally small circle around that point. If the field is conservative, then the work done over *any* closed curve is zero and so the curl at any point should be zero.