16.2-3: More Line Integrals

Monday, April 18

Generic Line Integral

(16.2.15) Find $\int_C z^2\,dx + x^2\,dy + y^2\,dz$ where C is the line segment from $(1,0,0)$ to $(4,1,2)$. Parametrize with $x = 3t+1, y = t, z = 2t$ as t goes from 0 to 1. Then get the integral as

$$\begin{aligned}\int_C \langle z^2, x^2, y^2\rangle \cdot \langle 3,1,2\rangle\,dt &= \int_{t=0}^{1} \langle 3(2t)^2 + (3t+1)^2 + 2t^2\,dt \\ &= [4t^3 + (3t^3 + 3t^2 + t) + \frac{2}{3}t^3]_{t=0}^{1} \\ &= [4 + 7 + 2/3] \\ &= 35/3.\end{aligned}$$

Work

An cannonball of mass m is thrown at an angle α and initial velocity v_0. Calculate the amount of work gravity does on the cannonball (constant downward force mg) from the time it is fired until the time it hits the ground.

Easy answer: zero, because the force is conservative and the object starts and ends at ground level. This may seem slightly nonintuitive because the gravity *does* change the object's momentum, but that's a different matter.

Regular answer: get $x(t) = v_0 t\cos\alpha$, $y(t) = v_0 t\sin\alpha - \frac{1}{2}gt^2$. Solve for $y = 0$ to get the end time $t = 2v_0\sin\alpha/g$, so the work is

$$\begin{aligned}\int_{t=0}^{2v_0\sin\alpha/g} \langle 0, -mg\rangle \cdot \langle v_0\cos\alpha, v_0\sin\alpha - gt\rangle\,dt &= \int_{t=0}^{2v_0\sin\alpha/g} -v_0 mg\sin\alpha + mg^2 t\,dt \\ &= [\frac{1}{2}mg^2t^2 - v_0 mgt\sin\alpha]_{t=0}62v_0\sin\alpha/g \\ &= [2mv_0^2\sin^2\alpha - 2mv_0^2\sin^2\alpha] \\ &= 0.\end{aligned}$$

(16.2.47)

1. Show that a constant force field does no work on a particle that moves once around the circle $x^2+y^2=1$.

   Easy: it's conservative.

   Regular: $x=\cos t, y=\sin t$, $F(x,y)=\langle a,b\rangle$, so $\displaystyle\int_{t=0}^{2\pi} F\cdot d\mathbf{r} = \int_{t=0}^{2} \pi - a\sin t + b\cos t\,dt = 0$.

2. Is this also true for the force field $\mathbf{F}(x,y)=k\langle x,y\rangle$?

   Yes. Easy answer: it's conservative, since $\langle x,y\rangle$ is the gradient of x^2+y^2.

   Regular answer: $\langle x,y\rangle\cdot\langle x',y'\rangle = \langle\cos t,\sin t\rangle\cdot\langle -\sin t,\cos t\rangle = 0$, so at no point is the force field doing any work.

3. How much work will the above fields do if the particle only moves around part of a circle?

   It depends for the first field, but will always be zero for the second.

True or False?

1. $\int_{-C} f\,ds = -\int_C f\,ds$: very FALSE. Not to be confused with the work done on a particle moving one direction along a curve versus moving in the opposite direction.
2. If a particle travels in a closed loop then the total work done on the particle over the loop is zero. FALSE: the force may not be conservative (a racecar can accelerate on a circular track... non-zero work done)
3. If we have a region S in (u,v)-space and form a region R by the transformation $x=2u+v, y=u-2v$, then the area of R is 1/5 the area of S. FALSE: it's 5 times the area.
4. If we instead apply the transformation $x=2u+v, y=4u+2v$ then the region of R is zero. TRUE
5. If a particle is moving in a constant force field then the work done on the particle is proportional to the distance the object travels. FALSE
6. If a particle is moving in a constant force field then the work done on the particle is proportional to the particle's distance from its starting position. FALSE: only it's displacement in the direction parallel to the force matters, no movement normal to the force matters.