15.8-15.9: Triple Integrals, Change of Variables

Monday, April 11

Spherical Coordinates

Find the volume of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere with boundary $x^2 + y^2 + z^2 = 1$. What is the ratio of this volume to the volume of the sphere? Make an estimate before finding the answer.

The spherical boundary gives the bounds $0 \le \rho \le 1$ and rotational symmetry around the z-axis gives the bounds $0 \le \theta \le 2\pi$, so we just need to find the upper bound on ϕ given by the cone.
You could either have this memorized (the standard cone makes a 45 degree angle with the xy plane), or substitute $x^2 + y^2 = r^2$. From here, the graph $z = |r|$ makes 45 degree angles with the r-axis, so you could stop there. If you want to expand everything in terms of spherical coordinates, get $z = \rho \cos \phi$ and $r = \rho \sin \phi$, so $z^2 \ge r^2$ implies $\rho^2 \cos^2 \phi \ge \rho^2 \sin^2 \phi$, which means that $1 \ge \tan(\phi)$. This is the most general technique, but it might be simplest just to remember the standard cone since you'll be most likely to see that one (or you'll be provided the angle for the cone directly).
The volume is therefore

$$\int_{\rho=0}^{1} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \left(\int \rho^2 \right) \left(\int d\theta \right) \left(\int \sin \phi \right)$$
$$= \frac{1}{3}(2\pi)(1 - \sqrt{2}/2)$$
$$= \frac{\pi}{3}(2 - \sqrt{2}),$$

which is a little less than 1/6 the volume of the whole sphere.

Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.
If you let ρ and ϕ vary freely and integrate over θ, you will be integrating over orange wedges (or watermelon slices, whichever you prefer). It is therefore most natural to set this up so that $0 \le \rho \le 1$ and $0 \le \phi \le \pi$, but with the boundaries $0 \le \theta \le \pi/6$. The integral will then be $\frac{1}{9}\pi a^3$, or 1/12 the volume of the whole sphere (since $\pi/6$ is 1/12 of the full 2π).

True or False?

1. If a particle moves around on the surface of a sphere with $d\phi/dt$ and $d\theta/dt$ constant, then the speed of the particle is constant.

   FALSE, since $d\theta/dt$ being constant means that the particle will move faster when it is near the equator than when it is near one of the poles.

2. If a particle has fixed coordinates ϕ and θ but moves with $d\rho/dt$ constant, then the speed of the particle is constant.

   TRUE, since the particle is moving on a straight line from the origin it speed is equal to $|d\rho/dt|$.

3. $\displaystyle\int_{y=1}^{4} \int_{x=0}^{1} (x^2 + \sqrt{y}) \sin(x^2 y^2) \, dx \, dy \le 9.$

   TRUE: in general if $f(x,y) \le K$ and a domain D has area A, $\iint_D f(x,y) \, dx \, dy \le K \cdot A$. Here, the domain is a rectangle with area 3, so the trick is to show that $(x^2 + \sqrt{y}) \sin(x^2 y^2) \le 3$ for all (x, y) in the rectangle.

4. Every point in $\mathbb{R}^3$ is *uniquely* represented by a set of spherical coordinates (ρ, θ, ϕ).

   FALSE: in particular, when $\rho = 0$ the point will be the origin regardless of θ and ϕ.

5. $\int_0^1 \int_0^x \sqrt{x + y^2}\, dy\, dx = \int_0^x \int_0^1 \sqrt{x + y^2}\, dx\, dy$

   FALSE: The second integral isn't even well-defined on account of the $\int_0^x$ term! You have to be more careful when changing the limits of integration and make sure that your new limits specify the same geometric domain as the old ones.

6. When $f(x, y, z) = 1$, the integral $\iiint_V f(x, y, z)\, dx\, dy\, dz$ gives the volume of the region V.

   TRUE.

Change of Variables

Evaluate the following integral by making the change of coordinates $u = 3x, v = 2y$:

$$\iint_R \sin(9x^2 + 4y^2)\, dA$$

where R is the region in the first quadrant bounded by the ellipse $9x^2 + 4y^2 = 1$.
First, $x = u/3$ and $y = v/2$, so $\frac{\partial(x,y)}{\partial u, v} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$. Therefore,

$$\iint_R \sin(9x^2 + 4y^2)\, dA = \iint_S \sin(u^2 + v^2) \begin{vmatrix} 1/3 & 0 \\ 0 & 1/2 \end{vmatrix} dA,$$

where S is the unit circle $u^2 + v^2 \leq 1$. Since the domain is a nice circle, we can change the system to polar coordinates with $u = r\cos\theta, v = r\sin\theta$:

$$\begin{aligned} \frac{1}{6} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r \sin(r^2)\, d\theta\, dr &= \frac{1}{3}\pi[-\frac{1}{2}\cos(r^2)]|_0^1 \\ &= \frac{1}{6}\pi(1 - \cos 1). \end{aligned}$$