Problem 1

Determine, with proof, whether each of the following functions is continuous at the origin:

1. \( f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases} \)

2. \( g(x, y) = \begin{cases} \frac{2xy^2}{x^4+y^4} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases} \)
Problem 2
A particle is at position \( (3.02, 1.97, 5.99) \). Use a linear approximation to estimate its distance from the origin.

Problem 3
Find the maximum and minimum values of the function \( f(x, y) = x^2 + 2xy - 2x - 2y + y^2 \) given the constraints \( x^2 + y^2 = 1 \).
Problem 4
Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

Problem 5
Find and classify all critical points of the function $f(x, y) = x^3 + xy + y^2$. 
True or False?

1. If a function $f$ has a single global maximum at $(a, b)$ then $\nabla f(x, y)$ points along the line segment from $(x, y)$ to $(a, b)$.

2. For any unit vector $u$ and any point $a$, $Df_{-u}(a) = -Df_u(a)$.

3. If $f_x$ and $f_y$ exist and are continuous in a neighborhood around $(a, b)$ then $f$ is differentiable at $(a, b)$.

4. If $f$ has a unique global maximum at a point $a$ then the maximum value of $f$ on a domain $D$ occurs at the point in $D$ closest to $a$.

5. There exists a function $f$ with continuous second-order partial derivatives such that $f_x(x, y) = x + y^2$ and $f_y(x, y) = x - y^2$.

6. $f_y(a, b) = \lim_{y \to b} \frac{f(a, y) - f(a, b)}{y - b}$.

7. If $f$ and $g$ are both differentiable, then $\nabla(fg) = f\nabla g + g\nabla f$.

8. If $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some $\lambda$ then $x$ is an extreme value of $f$ on the set $\{(a, b) : g(a, b) = g(x, y)\}$.

9. If $f(x, y) = f(y, x)$ for all $x, y \in \mathbb{R}$ then $\int_0^a \int_0^b f(x, y) \, dy \, dx = \int_0^b \int_0^a f(x, y) \, dy \, dx$.

10. For any integrable function $f$, $\int_0^a \int_0^b f(x, y) \, dx \, dy = \int_0^b \int_0^a f(x, y) \, dx \, dy$. 