Midterm 2: Practice Test

Monday, April 4

Problem 1

Determine, with proof, whether each of the following functions is continuous at the origin:

1. $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

   This function is NOT continuous at the origin. If we approach it along the line $y = 0$ or $x = 0$ then the limit will be zero, but along the line $y = x$ the limit is 1.

2. $g(x,y) = \begin{cases} \frac{2xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

   This function is continuous at the origin. The simplest route: since $y^2 \leq x^2 + y^2$, $|\frac{2xy^2}{x^2+y^2}| \leq |2x|$ for all points (x,y). Since $\lim_{(x,y)\to 0} |2x| = 0$, the Squeeze theorem implies that the limit of g is also zero.

   A second method: make the substitution $x = r\cos\theta, y = r\sin\theta$ and get $g(x,y) = 2r^3\sin^2\theta\cos^2\theta/r^2 = 2r\sin^2\theta\cos^2\theta$, which approaches zero as $r \to 0$.

Problem 2

A particle is at position $\langle 3.02, 1.97, 5.99\rangle$. Use a linear approximation to estimate its distance from the origin. If $D(x,y,z) = \sqrt{x^2+y^2+z^2}$, then $\nabla D(x,y,z) = \frac{1}{D(x,y,z)}\langle x,y,z\rangle$. If we approximate from the point $\langle 3,2,6\rangle$ and set $h = \langle \Delta x, \Delta y, \Delta z\rangle = \langle 0.02, -0.03, -0.01\rangle$, then we get

$$D(3.02, 1.97, 5.99) \approx D(3,2,6) + \nabla D \cdot h = 7 + \frac{1}{7}\langle 3,2,6\rangle \cdot \langle 0.02, -0.03, -0.01\rangle = 7 - 0.06/7.$$

Problem 3

Find the maximum and minimum values of the function $f(x,y) = x^2+2xy-2x-2y+y^2$ given the constraints $x^2+y^2 = 1$.

Use Lagrange multipliers: if $g(x,y) = x^2+y^2$, then $f(x,y) = \lambda g(x,y)$ gives us the equations

$$\begin{aligned} 2x + 2y - 2 &= 2\lambda x \\ 2x + 2y - 2 &= 2\lambda y \\ x^2 + y^2 &= 1. \end{aligned}$$

The left-hand sides of the first two equations are equal, implying that $\lambda x = \lambda y$. Therefore either $\lambda = 0$ or $x = y$. If $\lambda = 0$ then $x + y = 1$, and together with the constraint $x^2 + y^2 = 1$ it follows that $(1,0)$ and $(0,1)$ are possible solutions. In either case $f(x,y) = -1$.
If $\lambda \neq 0$ then $x = y$, which when paired with the constraint $x^2+y^2 = 1$ gives the solutions $\pm(\sqrt{2}/2, \sqrt{2}/2)$, in which case $f(x,y) = 2 \mp 2\sqrt{2}$.
Therefore, f hits a minimum of -1 at $(1,0)$ and $(0,1)$ and a maximum of $2+2\sqrt{2}$ at $(-\sqrt{2}/2, -\sqrt{2}/2)$.

Problem 4

Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

Polar coordinates will be the simplest way to set up the problem, but first we have to find where the cone and sphere intersect: if $z = \sqrt{x^2 + y^2}$ then $z^2 = x^2 + y^2$, so the two shapes intersect whenever $x^2 + y^2 = 1/2$. This makes the domain a circle with radius $r = \sqrt{2}/2$.
Then at any point (r, θ) the vertical distance from the sphere to the cone is $\sqrt{1 - r^2} - r$, so compute the volume as follows:

$$\begin{aligned} V &= \int_{r=0}^{\sqrt{2}/2} \int_{\theta=0}^{2\pi} (\sqrt{1 - r^2} - r) r \, d\theta \, dr \\ &= 2\pi \int_{r=0}^{\sqrt{2}/2} r\sqrt{1 - r^2} - r^2 \, dr \\ &= 2\pi [-\frac{1}{3}(1 - r^2)^{3/2} - \frac{r^3}{3}]_0^{\sqrt{2}/2} \\ &= 2\pi (\frac{-1}{3} \frac{\sqrt{2}4}{-} \frac{1}{3} \frac{\sqrt{2}}{4} + \frac{1}{3}) \\ &= \frac{\pi}{3}(2 - \sqrt{2}). \end{aligned}$$

Problem 5

Find and classify all critical points of the function $f(x, y) = x^3 + xy + y^2$.

$\nabla f(x, y) = \langle 3x^2 + y, x + 2y \rangle$, so setting the gradient to zero implies $y = -x/2$, and so $3x^2 - x/2 = x(3x - 1/2) = 0$. This gives the two solutions $x = 0$ and $x = 1/6$, so the only two critical points occur at $(0, 0)$, and $(1/6, -1/12)$.
The second partial derivatives of f are $\begin{bmatrix} 6x & 1 \\ 1 & 2 \end{bmatrix}$, so the Second Derivative Test implies that the point $(0, 0)$ is a saddle point and $(1/6, -1/12)$ is a local minimum.

True or False?

1. If a function f has a single global maximum at (a, b) then $\nabla f(x, y)$ points along the line segment from (x, y) to (a, b). FALSE: the gradient points in the direction of steepest ascent, which is not necessarily directly toward the global maximum. (For example, most points on a non-circular ellipse)

2. For any unit vector $\mathbf{u}$ and any point $\mathbf{a}$, $Df_{-\mathbf{u}}(\mathbf{a}) = -Df_{\mathbf{u}}(\mathbf{a})$. TRUE, since $\nabla f \cdot (-\mathbf{u}) = -\nabla f \cdot \mathbf{u}$ at any point for any vector.

3. If f_x and f_y exist and are continuous in a neighborhood around (a, b) then f is differentiable at (a, b). TRUE

4. If f has a unique global maximum at a point $\mathbf{a}$ then the maximum value of f on a domain D occurs at the point in D closest to $\mathbf{a}$. FALSE: say. $f(x, y) = -10x^2 - y^2$ where D is the line $y = 1 - x$.

5. There exists a function f with continuous second-order partial derivatives such that $f_x(x, y) = x + y^2$ and $f_y(x, y) = x - y^2$. FALSE, since we would have $f_{xy}(x, y) = 2y$ but $f_{yx}(x, y) = 1$.

6. $f_y(a, b) = \lim_{y \to b} \frac{f(a, y) - f(a, b)}{y - b}$. TRUE.

7. If f and g are both differentiable, then $\nabla(fg) = f\nabla g + g\nabla f$. TRUE.

8. If $\nabla f(x,y) = \lambda \nabla g(x,y)$ for some λ then x is an extreme value of f on the set $\{(a,b) : g(a,b) = g(x,y)\}$. FALSE: it could be a saddle point (for example, if $g(x,y) = 0$ for all (x,y) then this is just the same as finding a critical point of f.)

9. If $f(x,y) = f(y,x)$ for all $x, y \in \mathbb{R}$ then $\int_{x=0}^{a} \int_{y=0}^{b} f(x,y)\,dy\,dx = \int_{x=0}^{b} \int_{y=0}^{a} f(x,y)\,dy\,dx$. TRUE

10. For any integrable function f, $\int_{x=0}^{a} \int_{y=x}^{a} f(x,y)\,dx\,dy = \int_{y=0}^{a} \int_{x=y}^{a} f(x,y)\,dx\,dy$. FALSE: the two integrals describe different triangular domains.