15.2-3: Recap

Monday, March 28

Optimization

Find the point(s) in the region $\{(x, y) : x^2 \leq y \leq 4\}\ldots$

1. closest to the point $(0, 1)$.

   If the distance from $(0, 1)$ is $D(x, y) = \sqrt{x^2 + (y-1)^2}$, then we can equivalently minimize $D^2 := x^2 + (y-1)^2$. The gradient is zero at $(0, 1)$, which is in the domain.

2. closest to the point $(3, 0)$.

   Like the previous problem, we can say $D^2(x, y) = (x-3)^2 + y^2$, and the gradient is zero only at $(3, 0)$. This is not inside the domain, so the minimum appears on the border instead. Along the lower boundary $y = x^2$, we get $D^2(x) = (x-3)^2 + x^4$. The derivative is $4x^3 + 2(x-3) = 4x^3 + 2x - 6 = 2(x-1)(2x^2 + 2x + 3)$, which is zero only when $x = 1$. Skip checking the top border by saying that the point $(1, 1)$ is intuitively the closest (rather than the furthest).

3. furthest from the origin.

   Checking the boundaries $y = x^2$ and $y = 4$ show that the minimum does not occur along the interior of either boundary, so it must be at one of the vertices where the boundaries meet. The two points $(2, 4)$ and $(-2, 4)$ are equally far from the origin and these are the farthest of any in the domain.

Order of Integration

Set up a polar double integral in (r, θ) to find the volume of a cone of height h and radius R. If you integrate over r first, what does the remaining 1-dimensional integral represent? What if you integrate over θ first? Make some sketches.
$\int_{r=0}^{R} \int_{\theta=0}^{2\pi} h - \frac{h}{R} r \cdot r \, d\theta \, dr = \int_{r=0}^{R} 2\pi r h (1 - \frac{r}{R}) \, dr$, which is integration by cylindrical shells.
$\int_{\theta=0}^{2\pi} \int_{r=0}^{R} h(1 - \frac{r}{R}) r \, dr \, d\theta = \int_{\theta=0}^{2} \pi h R^2/6 \, d\theta$. This is a bit harder to interpret, but roughly the $hR^2 \Delta\theta/6$ is the volume of a triangular pyramid with height h and base side lengths R, R, and $\Delta\theta$.

Double Integrals

Sketch each given domain and set up an appropriate double integral $\iint_D f(x,y)$ on the domain. Then find the integral.

1. $D = \{(x,y) : x^2 + y^2 \leq 1, y \leq x\}, f(x,y) = e^{x^2+y^2}$

   The domain is a semicircle, so set up a polar integral: $\int_{r=0}^{1} \int_{\theta=-3\pi/4}^{\pi/4} f(x,y) r \, dr \, d\theta$. Then express f in terms of r and integrate.

2. $D = \{(x,y) : x - 5 \leq y \leq 1 - x^2\}, f(x,y) = x - 2y$

   The domain is above a line but below a downward-facing parabola, so find the points of intersection: $(-3,-8)$ and $(2,-3)$. It's easier to express the bounds of y in terms of those on x, so get $\int_{x=-3}^{2} \int_{y=x-5}^{1-x^2} f(x,y) \, dx \, dy$. Then solve.

3. $D = \{(x,y) : 2y^2 \leq x \leq 1 + y^2\}, f(x,y) = xy - 1$

   This was an example in the book, except with x and y switched. The domain is sandwiched between two parabolas, so find the points of intersection: $(1,1)$ and $(1,-1)$. It's easier to express the bounds of x in terms of y, so get $\int_{y=-1}^{1} \int_{x=2y^2}^{1+y^2} f(x,y) \, dx \, dy$. Then solve.

4. Given a cone of uniform density with radius R and height h, find the smallest r such that at least half of the cone's mass is within distance r of its axis of symmetry.

   This is really a one-dimensional integration problem: the volume of the cone is $\frac{1}{3}\pi h R^2$, so we want to find the value M such that $\frac{1}{6}\pi h R^2 = \int_{r=0}^{M} 2\pi r h (1 - \frac{r}{R}) \, dr$.

   This simplifies to finding M such that $\frac{1}{12}R^2 = M^2/2 - M^3/3R$. Simplify this by saying $M = cR$ to get $1/2 = 3c^2 - 2c^3$, which has the solution $c = 1/2$. Is there some reason that this answer should be intuitive?