Review 2: Many True/False

Wednesday, May 4

1. If $\mathbf{F}$ is a vector field then $\nabla \cdot \mathbf{F}$ is a vector field. FALSE: the divergence is a scalar function.
2. If $\mathbf{F}$ is a vector field then $\nabla \times \mathbf{F}$ is a vector field. TRUE.
3. I f has continuous partial derivatives on $\mathbb{R}^3$ then $\nabla \cdot (\nabla \times f) = 0$. TRUE.
4. If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle then $\int_C \nabla f \cdot d\mathbf{r} = 0$. FALSE: this is necessarily true only if the curl of f is zero.
5. If $\mathbf{F} = \langle P, Q \rangle$ and $P_y = Q_x$ in an open region D then $\mathbf{F}$ is conservative. FALSE: D must be simply connected.
6. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields and $\nabla \times \mathbf{F} = \nabla \times \mathbf{G}$ then $\mathbf{F} = \mathbf{G}$. FALSE: F can be G plus any function whose curl is zero.
7. The work done by a conservative force field in moving a particle around a closed path is zero. TRUE.
8. There is a vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F}\langle x, y, z \rangle$. FALSE: this function has non-zero divergence, but an earlier true/false implies that the divergence of the curl of any smooth function is zero.
9. If $\mathbf{F}$ is conservative then $\nabla \times \mathbf{F} = 0$: TRUE
10. If $\mathbf{F}$ is conservative then $\nabla \cdot \mathbf{F} = 0$: FALSE. This would imply that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ (or analogously in any number of dimensions), but lots of functions (e.g. $f(x, y) = x^2 + y^2$) violate this equality.
11. If $\nabla \times \mathbf{F} = 0$ then $\mathbf{F}$ is conservative: FALSE. This was true as long as $\mathbf{F}$ is defined on all of $\mathbb{R}^3$.
12. Green's Theorem is just the Divergence Theorem in two dimensions. FALSE: it's Stokes' Theorem in two dimensions.
13. curl(div($\mathbf{F}$)) is not a meaningful expression. TRUE, since curl must take a 3-D vector field as its argument, but div(F) is a scalar.
14. The integral $\displaystyle\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{1} \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$ gives the volume of 1/4 of a sphere. FALSE for two reasons: the $\sin\theta$ should be $\sin\phi$, and even with this change it only gives the volume of 1/8 of a sphere.
15. $\displaystyle\int_{r=-1}^{1} \int_{\theta=0}^{1} e^{r^2+\theta^2} \, d\theta \, dr = \left[\int_{r=-1}^{1} e^{r^2} \, dr \right] \left[\int_{\theta=0}^{1} e^{\theta^2} \, d\theta \right]$: TRUE.
16. If C is a closed curve then $\int_C f \, ds = 0$ for any function f: Very FALSE.
17. If $\int_C f \, ds = 0$ then C is a closed curve. FALSE: for example, the integral of $\sin x$ as x goes from 0 to 2π.
18. If the work done by a force $\mathbf{F}$ on an object moving along a curve is W, then if the object moves along the curve in the opposite direction the work done by $\mathbf{F}$ will be $-W$. TRUE.
19. If a particle moves along a curve C, the total work done by a force $\mathbf{F}$ on the object is independent of how quickly the particle moves. TRUE.
20. If a force points only in the x direction then the work done by the force on a particle depends only on the particle's starting and ending x-positions. FALSE: the force's strength could depend on the y-position of the particle (Imagine swimming up a river versus walking along the bank. The river's current does different amounts of work.)

21. If ∇f exists everywhere then f is continuous everywhere. TRUE: if a function is differentiable it must be continuous (but not the other way round!)

22. If $f_x = f_y = 0$ at a point (x, y) then f is differentiable at (x, y). FALSE: it might not even be continuous! (come up with examples)

23. If f is differentiable along every straight line going through a point (x, y) then f is differentiable at (x, y). FALSE: Still might not even be continuous!

24. For any x, $f(x - \nabla f(x)) \leq f(x)$. FALSE: the negative gradient is a descent direction, so what's true is that if $\nabla f \neq 0$ then there exists $\lambda > 0$ (possibly very small) such that $f(x - \lambda \nabla f(x)) < f(x)$.

25. If $f(x, y) = 1$ then $\iint_D f(x, y)\, dA$ is equal to the area of the domain D. TRUE

26. For any $a, b \in \mathbb{R}$ and continuous function f, $\int_{x=0}^{a} \int_{y=0}^{b} f(x, y)\, dy\, dx = \int_{y=0}^{b} \int_{x=0}^{a} f(x, y)\, dx\, dy$. TRUE

27. If $f(x, y) = g(x)h(y)$, then $\iint_D f(x, y)\, dA = \left(\iint_D g(x)\, dA\right)\left(\iint_D h(y)\, dA\right)$. FALSE: you can split the integral as a product of two single-variable integrals if the integral is over a rectangle.

28. If $f_{xx} > 0$ and $f_{yy} > 0$ at a point (x, y) then the point (x, y) is a local minimum of the function f. FALSE: if f_{xy}, f_{yx} are large then it could be a saddle point.

29. If (x, y) is a local minimum of a function f then f is differentiable at (x, y) and $\nabla f(x, y) = 0$. FALSE: say, $f(x, y) = |x| + |y|$.

30. If $\nabla f(x, y) = 0$ then (x, y) is a local minimum or maximum of f. FALSE: it could be a saddle point.

31. If $f_{xx} > 0$ and $f_{yy} < 0$ at a point (x, y) then (x, y) is a saddle point of f. TRUE, IF (x, y) is a critical point. Otherwise it's definitely false.

32. If ∇f is never zero then the minimum and maximum of f on a closed and bounded domain D must occur on the boundary. TRUE, assuming f is continuous and differentiable.

33. If f has a critical point in the interior of a closed and bounded domain D then the minimum and maximum of f on D occur in the interior of D. FALSE. (e.g. $f(x, y) = x^2 - y^2$ on the unit circle)

34. If x is a minimum of f given the constraints $g(x) = h(x) = 0$ then $\nabla f(x) = \lambda \nabla g(x)$ and $\nabla f(x) = \mu \nabla h(x)$ for some scalars λ and μ.

    FALSE: $\nabla f(x) = \lambda \nabla g(x) + \mu \nabla h(x)$ for some λ, μ.

35. A region D simply connected if any two points in D can be joined by a curve that stays inside D. FALSE: this only describes a connected region (mostly). D must also not have any "holes," so a doughnut shape would be a counterexample.

36. If $\mathbf{F}$ is conservative on D and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ (where P and Q are continuously differentiable) then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D. TRUE.

37. If P and Q are continuously differentiable and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D then $\mathbf{F}$ is conservative on D. FALSE: the domain must be simply connected.

38. If there exists a closed curve C in D such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ then $\mathbf{F}$ is conservative on D. FALSE: the condition is that this holds for *all* curves in D.

39. If $\mathbf{F}$ is conservative on a region D then there is some function f on D such that $\nabla f = \mathbf{F}$. TRUE.

40. If two triangles share an edge then the work a force field does on a particle traveling along the two triangles one at a time is the same as if the particle traveled along the quadrilateral boundary of the union of the two triangles. FALSE: only if the particle travels along both triangles in the same direction, in which case it's true (the integrals over the shared edge have to cancel out).

41. The kinetic energy of an object plus its potential energy due to gravity is always constant: depends on whether you assume that gravity is the only thing doing work on an object. If yes, then true, but I would say in general FALSE.

42. $\int_{-C} f\,ds = -\int_C f\,ds$: very FALSE. Not to be confused with the work done on a particle moving one direction along a curve versus moving in the opposite direction.

43. If a particle travels in a closed loop then the total work done on the particle over the loop is zero. FALSE: the force may not be conservative (a racecar can accelerate on a circular track... non-zero work done)

44. If we have a region S in (u,v)-space and form a region R by the transformation $x = 2u + v, y = u - 2v$, then the area of R is 1/5 the area of S. FALSE: it's 5 times the area.

45. If we instead apply the transformation $x = 2u + v, y = 4u + 2v$ then the region of R is zero. TRUE

46. If a particle is moving in a constant force field then the work done on the particle is proportional to the distance the object travels. FALSE

47. If a particle is moving in a constant force field then the work done on the particle is proportional to the particle's distance from its starting position. FALSE: only it's displacement in the direction parallel to the force matters, no movement normal to the force matters.

48. If a particle moves around on the surface of a sphere with $d\phi/dt$ and $d\theta/dt$ constant, then the speed of the particle is constant.

    FALSE, since $d\theta/dt$ being constant means that the particle will move faster when it is near the equator than when it is near one of the poles.

49. If a particle has fixed coordinates ϕ and θ but moves with $d\rho/dt$ constant, then the speed of the particle is constant.

    TRUE, since the particle is moving on a straight line from the origin it speed is equal to $|d\rho/dt|$.

50. $$\int_{y=1}^{4}\int_{x=0}^{1}(x^2+\sqrt{y})\sin(x^2y^2)\,dx\,dy \leq 9.$$

    TRUE: in general if $f(x, y) \leq K$ and a domain D has area A, $\iint_D f(x, y)\,dx\,dy \leq K \cdot A$. Here, the domain is a rectangle with area 3, so the trick is to show that $(x^2 + \sqrt{y})\sin(x^2y^2) \leq 3$ for all (x, y) in the rectangle.

51. Every point in $\mathbb{R}^3$ is *uniquely* represented by a set of spherical coordinates (ρ, θ, ϕ).

    FALSE: in particular, when $\rho = 0$ the point will be the origin regardless of θ and ϕ.

52. $$\int_0^1\int_0^x \sqrt{x+y^2}\,dy\,dx = \int_0^x\int_0^1 \sqrt{x+y^2}\,dx\,dy$$

    FALSE: The second integral isn't even well-defined on account of the $\int_0^x$ term! You have to be more careful when changing the limits of integration and make sure that your new limits specify the same geometric domain as the old ones.

53. When $f(x, y, z) = 1$, the integral $\iiint_V f(x, y, z)\,dx\,dy\,dz$ gives the volume of the region V.

    TRUE.

54. If a function f has a single global maximum at (a, b) then $\nabla f(x, y)$ points along the line segment from (x, y) to (a, b). FALSE: the gradient points in the direction of steepest ascent, which is not necessarily directly toward the global maximum. (For example, most points on a non-circular ellipse)

55. For any unit vector $\mathbf{u}$ and any point $\mathbf{a}$, $Df_{-\mathbf{u}}(\mathbf{a}) = -Df_{\mathbf{u}}(\mathbf{a})$. TRUE, since $\nabla f \cdot (-\mathbf{u}) = -\nabla f \cdot \mathbf{u}$ at any point for any vector.

56. If f_x and f_y exist and are continuous in a neighborhood around (a, b) then f is differentiable at (a, b). TRUE

57. If f has a unique global maximum at a point $\mathbf{a}$ then the maximum value of f on a domain D occurs at the point in D closest to $\mathbf{a}$. FALSE: say. $f(x, y) = -10x^2 - y^2$ where D is the line $y = 1 - x$.

58. There exists a function f with continuous second-order partial derivatives such that $f_x(x, y) = x + y^2$ and $f_y(x, y) = x - y^2$. FALSE, since we would have $f_{xy}(x, y) = 2y$ but $f_{yx}(x, y) = 1$.

59. $f_y(a, b) = \lim_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$. TRUE.

60. If f and g are both differentiable, then $\nabla(fg) = f\nabla g + g\nabla f$. TRUE.

61. If $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some λ then x is an extreme value of f on the set $\{(a, b) : g(a, b) = g(x, y)\}$. FALSE: it could be a saddle point (for example, if $g(x, y) = 0$ for all (x,y) then this is just the same as finding a critical point of f.)

62. If $f(x, y) = f(y, x)$ for all $x, y \in \mathbb{R}$ then $\displaystyle\int_{x=0}^{a} \int_{y=0}^{b} f(x, y)\, dy\, dx = \int_{x=0}^{b} \int_{y=0}^{a} f(x, y)\, dy\, dx$. TRUE

63. For any integrable function f, $\displaystyle\int_{x=0}^{a} \int_{y=x}^{a} f(x, y)\, dx\, dy = \int_{y=0}^{a} \int_{x=y}^{a} f(x, y)\, dx\, dy$. FALSE: the two integrals describe different triangular domains.

64. If $f_x(a, b)$ and $f_y(a, b)$ both exist then f is differentiable at (a, b): FALSE— just because two directional derivatives exists doesn't mean the function is differentiable. (recall the function $f(x, y) = xy^2/(x^2 + y^4)$)

65. If $f(x, y) = \ln y$ then $\nabla f(x, y) = 1/y$. FALSE: $\nabla f(x, y) = \langle 0, 1/y \rangle$.

66. If f has a local minimum at (a, b) and f is differentiable at (a, b) then $\nabla f(a, b) = 0$. TRUE

67. If $f(x, y) = \sin x + \sin y$ then $-\sqrt{2} \leq D_u f(x, y) \leq \sqrt{2}$ for all unit vectors u. TRUE, since $|D_u f(x, y)| = |\nabla f(x, y) \cdot u| \leq |\nabla f(x, y)| \cdot |u| = |\nabla f(x, y)|$.

68. If f is differentiable at (a, b) and $\nabla f(a, b) = 0$ then f has a local maximum or minimum at (a, b). FALSE. It could be a saddle point.

69. If $\nabla f(a, b) = 0$, $f_{xx}(a, b) > 0$ and $f_{yy}(a, b) > 0$, then f has a local minimum at (a, b). FALSE: if f_{xy} is sufficiently large then the point could still be a saddle point. But (a, b) is definitely not a local maximum.

70. If $f(x, y)$ has two local maxima then f must have a local minimum. FALSE: It could just have a saddle point in between the maxima (imagine a mountain with two peaks: it doesn't have a local minimum elevation).

71. If f has a single global minimum at (a, b), then the minimum of f on the unit circle occurs at the point on the circle closest to (a, b). FALSE.

72. If $f(x,y) \to L$ as $(x,y) \to (a,b)$ along every straight line through (a,b) then $\lim_{(x,y)\to(a,b)} f(x,y) = L$. False.

73. If f is a function then $\lim_{(x,y)\to(2,5) f(x,y)} = f(2,5)$: False: this is true if f is continuous.

74. If $f(x,y)$ is continuous and we define $g_0(y) = f(0,y)$, then g is also continuous: true.

75. If $f(x,y)$ has no global maximum or minimum and $g(x) = f(0,x)$, then $g(x)$ also has no global maximum or minimum: False: try $f(x,y) = x^2 - y^2$.

76. For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

    Visual intuition: up to sign, yes, because (with the zero vector) the triple product describes the volume of a parallelepiped.

77. For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

    Nope. $i \times (i \times j) = i \times k = -j$, but $(i \times i) \times j = 0 \times j = 0$.

78. If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

    Nope. They can be perpendicular.

79. If $\mathbf{u} \times \mathbf{v} = 0$ then $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

    Nope. They can be parallel.

80. The intersection of two non-parallel planes is always a line.

    Yup.

81. The polar curves $r = 1 - \sin 2\theta$, $r = \sin 2\theta - 1$ have the same graph.

    True.

82. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then $\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dx^2}$.

    False.

83. The distance traveled by an object is equal to the integral of its velocity over time.

    False... it's the integral of speed over time.

84. For any vectors u and v in $\mathbb{R}^n$, $u + v = v + u$. True.

85. For any vectors u and v in $\mathbb{R}^n$, $|u + v| = |u| + |v|$. False, unless the vectors are pointing in the same direction.

86. The set of points $\{x, y, z | x^2 + y^2 = 1\}$ is a circle. False: it's an infinite cylinder.

87. The curve defined by any set of parametric equations $(x,y) = (f(t), g(t))$ can also be defined by an equation of the form $y = h(x)$.

    False: $x = \cos\theta, y = \sin\theta$ defines a circle, and y cannot be expressed as a function of x.

88. The curve defined by any equation of the form $y = h(x)$ can also be defined by a set of parametric equations $(x,y) = (f(t), g(t))$.

    True: set $x = t$ and $g(t) = h(x)$.

89. If $dy/dt = 0$ at some point on a curve then the tangent line at that point is horizontal.

    False: if dx/dt is also zero then the tangent line is not necessarily horizontal.

90. If a circle is parametrized as $(x, y) = (\cos t, \sin t)$, then for any t the angle between $(x(t), y(t))$ and the positive x-axis will be equal to t.

    True for circles, but not for ellipses.

91. If $f(\theta) = f(-\theta)$ for all θ, then the curve defined by $r = f(\theta)$ will have a vertical axis of symmetry.

    False: it will have a horizontal axis of symmetry.

92. If $f(\theta) = f(\theta + \pi)$ for all θ, then the curve defined by $r = f(\theta)$ will be unchanged when it is rotated by 180 degrees about the origin.

    True.

93. If the force on a particle is always perpendicular to the particle's velocity then the particle will never change speed. TRUE: the normal component of acceleration (which here is all of the acceleration) only affects the curvature, not the speed.

94. If the force on a particle is always parallel to the particle's velocity then the particle will never change direction: Depends on whether you count stopping and reversing direction as being in the "same" direction. It's still parallel to the original velocity vector, but I would say FALSE.