15.1-2: Double Integrals

Friday, March 18

Integrals over General Regions

Find the integral of the function $f(x, y) = xy^2$ over the triangle with vertices at $(0, 0), (0, 1)$, and $(1, 0)$ at least two different ways.

$$\begin{aligned}
\int_{x=0}^{1} \int_{y=0}^{1-x} xy^2 \, dy \, dx &= \frac{1}{3} \int_{x=0}^{1} xy^3|_0^{1-x} \, dx \\
&= \frac{1}{3} \int_{x=0}^{1} x(1-x)^3 \, dx \\
&= \frac{1}{3} \int_{0}^{1} x - 3x^2 + 3x^3 - x^4 \, dx \\
&= \frac{1}{3}(x^2/2 - x^3 + 3x^4/4 - x^5/5)|_0^1 \\
&= \frac{1}{3}(1/2 - 1 + 3/4 - 1/5) \\
&= \frac{1}{60}.
\end{aligned}$$

The second way,

$$\begin{aligned}
\int_{y=0}^{1} \int_{x=0}^{1-y} xy^2 \, dy \, dx &= \frac{1}{2} \int_{y=0}^{1} x^2y^2|_0^{1-y} \, dy \\
&= \frac{1}{2} \int_{0}^{1} y^2(1-y)^2 \, dy \\
&= \frac{1}{2} \int_{0}^{1} y^4 - 2y^3 + y^2 \, dy \\
&= \frac{1}{2}(y^5/5 - y^4/2 + y^3/3)|_0^1 \\
&= 1/60.
\end{aligned}$$

The two integrals are the same, as they should be.

Find the volume of the solid under the surface $z = xy$ and above the triangle with vertices $(1, 1), (4, 1), (2, 2)$.

This time the triangle isn't a right triangle, so if we integrated along the x-axis first we would likely have to split the integral in two. So integrate along the y-axis instead:

$$
\begin{aligned}
\int_{y=1}^{2}\int_{x=y}^{6-2y} xy\,dx\,dy &= \int_{y=1}^{2} \frac{y}{2}[(6-2y)^2 - y^2]\,dy \\
&= \frac{1}{2}\int_{y=1}^{2} (36y - 24y^2 + 3y^3)\,dy \\
&= \frac{1}{2}[18y^2 - 8y^3 + 3y^4/4]|_1^2 \\
&= \frac{1}{2}(72 - 64 + 12 - (18 - 8 + 3/4)) \\
&= 37/8.
\end{aligned}
$$

Evaluate the integral $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$ by reversing the order of integration.

The area being integrated over is a right triangle with vertices at $(0,0), (3,0), (3,1)$, so switch the order:

$$
\begin{aligned}
\int_0^1 \int_{x=3y}^{3} e^{x^2}\,dx\,dy &= \int_{x=0}^{3}\int_{y=0}^{x/3} e^{x^2}\,dx\,dy \\
&= \frac{1}{3}\int_{x=0}^{3} xe^{x^2}\,dx \\
&= \frac{1}{6}e^{x^2}|_0^3 \\
&= (e^9 - 1)/6.
\end{aligned}
$$

Polar Coordinates

Evaluate the integral $\iint_D e^{-x^2-y^2}\,dA$, where D is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and the y-axis.

$$
\begin{aligned}
\iint_D e^{-x^2-y^2}\,dA &= \int_{r=0}^{2}\int_{\theta=-\pi/2}^{\pi/2} e^{r^2}\, r\,dr\,d\theta \\
&= \pi\int_{r=0}^{2} re^{r^2}\,dr \\
&= \frac{\pi}{2}e^{r^2}|_0^2 \\
&= \frac{\pi}{2}(e^4-1).
\end{aligned}
$$

Find the volume of the solid inside the sphere $x^2+y^2+z^2 = 16$ and outside the cylinder $x^2+y^2 = 4$. Express this as twice the integral of the function $z = \sqrt{16-x^2-y^2}$ over the xy-plane. Writing in polar coordinates makes the inner radius $r = 2$ (from $x^2+y^2 \geq 2^2$) and the outer radius $r = 4$ (so $r^2 = 16$ and $z = 0$, the smallest value for which it is well-defined.) Therefore,

$$
\begin{aligned}
\int_{r=2}^{4}\int_{\theta=0}^{2\pi} \sqrt{16-r^2}r\,dr\,d\theta &= 2\pi\int_{\theta=0}^{2\pi} r\sqrt{16-r^2}\,dr \\
&= -\frac{2\pi}{3}(16-r^2)^{3/2}|2^4 \\
&= -\frac{2\pi}{3}(0-12^{3/2}) \\
&= \frac{2\pi}{3}8\cdot 3\sqrt{3} \\
&= 16\pi\sqrt{3}.
\end{aligned}
$$