14.5-6: Chain Rule, Partial Derivatives

Friday, March 4

Recap

Consider the function $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

Find f_x and f_y at the point $(0,0)$ and use it to approximate $f(0.1,-0.1)$. How good of an approximation is it to $f(0.1,-0.1)$?

f_x and f_y are both zero, so the linear approximation at $(0,0)$ is $f(x,y)=0$. Our approximation $f(0.1,-0.1) \approx 0$ is not that close to the true value of $-1/2$. This is because the function is not continuous at (0,0) let alone differentiable, so the linear approximation is close to worthless.

Do the same with the function $g(x,y) = x^2+y^2$. Why is the quality of your two approximations so different?

The linear approximation is again $g(x,y)=0$, but this time the function is differentiable and the approximation $g(0.1,-0.1) \approx 0$ is much closer to the true value of 0.02.

Chain Rule

Define $f(x,y) = ye^x$. If $y=t^2$ and $x=\sqrt{t}$, find $\frac{\partial f}{\partial t}$ when $t=1$.

$\frac{df}{dt} = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} = ye^x/2\sqrt{t} + e^x(2t)$. When $t=1$, $x=y=1$ and so this quantity is equal to $e/2+2e$.

If $g(x,y,z) = xyz$ and $x=t, y=t, z=t^2$, find $\frac{\partial g}{\partial t}$ at $t=2$. What does this have to do with the power rule? What about the product rule?

$$\begin{aligned} \partial g/\partial t &= \frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial t} \\ &= (yz)(1) + (xz)(1) + (xy)(2t) \\ &= 8+8+16 \\ &= 32. \end{aligned}$$

In general, if we set $f(x,y)=xy$ and $x=t, y=t$, then $\partial f/\partial t = xy' + x'y$, which is the product rule. If $g(t)=t^n$, then we can set $h(x_1,\ldots,x_n) = x_1x_2\cdots x_n$, and the chain rule gives $g'(t) = nt^{n-1}$.

Implicit Differentiation

Find $\partial z/\partial x$ and $\partial z/\partial y$ for an arbitrary point on the surface $x^2 + 2y^2 + 3z^2 = 1$.
Implicit differentiation gives $\frac{\partial z}{\partial x} = -3z/x$ and $\frac{\partial z}{\partial y} = -3z/2y$.

Do the same for the surface defined by the equation $e^z = xyz$.
Implicit differentiation with respect to x gives $e^z z' = x'yz + xyz'$, so $\frac{\partial z}{\partial x} = yz/(e^z - xy)$. The function is symmetric with respect to x and y, so $\frac{\partial z}{\partial y} = xz/(e^z - xy)$.

Directional Derivatives

Say we want to minimize the function $f(x, y) = x^2 + 2y^2 + xy + 7x$ and we are currently sitting at the point $(0, 0)$. Using the limit definition of a directional derivative, find the derivative of f in the direction $\mathbf{v} = \langle -\sqrt{2}/2, \sqrt{2}/2 \rangle$.

$$\begin{aligned} f_{\mathbf{v}}(0,0) &= \lim_{h \to 0} \frac{f((0,0) = h\mathbf{v}) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{f(-h\sqrt{2}2, h\sqrt{2}2)}{h} \\ &= \lim_{h \to 0} \frac{h^2/2 + h^2 - h^2/2 - 7h\sqrt{2}2}{h} \\ &= -7\sqrt{2}/2. \end{aligned}$$

Will the function decrease faster if we head in the direction $\langle -1, 1 \rangle$ or $\langle -3/5, 4/5 \rangle$?
The directional derivative in the direction $\langle -3/5, 4/5 \rangle$ is $-21/5$ which is smaller in magnitude than $-7\sqrt{2}/2$ (the directional derivative in the direction $\langle -1, 1 \rangle$), so the function will decrease faster going in the direction $\langle -1, 1 \rangle$.