Recap

Consider the function \( f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases} \). Find \( f_x \) and \( f_y \) at the point \((0,0)\) and use it to approximate \( f(0.1, -0.1) \). How good of an approximation is it to \( f(0.1, -0.1) \)?

\( f_x \) and \( f_y \) are both zero, so the linear approximation at \((0,0)\) is \( f(x, y) = 0 \). Our approximation \( f(0.1, -0.1) \approx 0 \) is not that close to the true value of \(-1/2\). This is because the function is not continuous at \((0,0)\) let alone differentiable, so the linear approximation is close to worthless.

Do the same with the function \( g(x, y) = x^2 + y^2 \). Why is the quality of your two approximations so different?

The linear approximation is again \( g(x, y) = 0 \), but this time the function is differentiable and the approximation \( g(0.1, -0.1) \approx 0 \) is much closer to the true value of 0.02.

Chain Rule

Define \( f(x, y) = ye^x \). If \( y = t^2 \) and \( x = \sqrt{t} \), find \( \frac{df}{dt} \) when \( t = 1 \).

\[
\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = ye^x/2\sqrt{t} + e^x(2t).
\]

When \( t = 1 \), \( x = y = 1 \) and so this quantity is equal to \( e/2 + 2e \).

If \( g(x, y, z) = xyz \) and \( x = t, y = t, z = t^2 \), find \( \frac{dg}{dt} \) at \( t = 2 \). What does this have to do with the power rule? What about the product rule?

\[
\frac{dg}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} + \frac{\partial g}{\partial z} \frac{dz}{dt} = (yz)(1) + (xz)(1) + (xy)(2t) = 8 + 8 + 16 = 32.
\]

In general, if we set \( f(x, y) = xy \) and \( x = t, y = t \), then \( \frac{df}{dt} = xy' + x'y \), which is the product rule. If \( g(t) = t^n \), then we can set \( h(x_1, \ldots, x_n) = x_1x_2 \cdots x_n \), and the chain rule gives \( g'(t) = nt^{n-1} \).
Implicit Differentiation

Find $\partial z/\partial x$ and $\partial z/\partial y$ for an arbitrary point on the surface $x^2 + 2y^2 + 3z^2 = 1$. Implicit differentiation gives $\frac{\partial z}{\partial x} = -3z/x$ and $\frac{\partial z}{\partial y} = -3z/y$.

Do the same for the surface defined by the equation $e^z = xyz$.
Implicit differentiation with respect to $x$ gives $e^z \frac{dz}{dx} = xy + x'y$, so $\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$. The function is symmetric with respect to $x$ and $y$, so $\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$.

Directional Derivatives

Say we want to minimize the function $f(x, y) = x^2 + 2y^2 + xy + 7x$ and we are currently sitting at the point $(0, 0)$. Using the limit definition of a directional derivative, find the derivative of $f$ in the direction $v = (-\sqrt{2}/2, \sqrt{2}/2)$.

$$f_v(0, 0) = \lim_{h \to 0} \frac{f((0, 0) + hv) - f(0, 0)}{h}$$
$$= \lim_{h \to 0} \frac{f(-h\sqrt{2}/2, h\sqrt{2}/2)}{h}$$
$$= \lim_{h \to 0} \frac{h^2/2 + h^2 - h^2/2 - 7h\sqrt{2}/2}{h}$$
$$= -7\sqrt{2}/2.$$ 

Will the function decrease faster if we head in the direction $(-1, 1)$ or $(-3/5, 4/5)$? The directional derivative in the direction $(-3/5, 4/5)$ is $-21/5$ which is smaller in magnitude than $-7\sqrt{2}/2$ (the directional derivative in the direction $(-1, 1)$), so the function will decrease faster going in the direction $(-1, 1)$.