14.1-2: Functions of Multiple Variables

Friday, February 26

Contour Plots

Sketch contour plots of the following functions. Locate local maxima and minima, or determine that there are none. Also sketch the graphs of the functions.

1. $f(x, y) = (x-3)^2 + 2(y+1)^2$: It's an elliptic paraboloid (elliptic level sets) with a minimum at $(3, -1)$.
2. $f(x, y) = x^2 - y^2$: It's a hyperbolic paraboloid (hyperbolic level sets, except for the level set 0 being two lines), and there is no maximum or minimum.
3. $f(\mathbf{x}) = \langle \mathbf{2}, \mathbf{1} \rangle \cdot \mathbf{x}$: It's a plane, and the level sets are lines perpendicular to $\langle 2, 1 \rangle$.
4. $f(x, y) = y/\sqrt{x^2 + y^2}$: If we wrote the function in polar coordinates it would be $f(r, \theta) = \sin \theta$. The level sets are V-shaped except for -1, and 1 and the function is discontinuous at $(x, y) = (0, 0)$.

The Unit Ball

Consider the three functions $f(x, y) = \max(|x|, |y|)$, $g(x, y) = \sqrt{x^2 + y^2}$, and $h(x, y) = |x| + |y|$. Sketch one or two level sets of the the three functions on the same plot. Come up with a conjecture about the relation between f, g, and h.

The level sets of f are squares, the level sets of g are circles, and the level sets of h are diamonds. The relation is $f(x, y) \leq g(x, y) \leq h(x, y)$ for all (x, y). (Prove it!)

Limits and Continuity

Find the limit or show that it does not exist:

1. $\lim_{(x,y)\to(1,2)} (5x^3 - x^2y^2)$

   The function is continuous and the limit is 1.

2. $\lim_{(x,y)\to(0.0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$

   If we go along the line $x = 0$ then the limit is -2, but if we go along the line $y = 0$ then the limit is zero. The limit therefore does not exist.

3. $\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$

   If we go along the line $y = x$ then the limit will be $1/2$, but if we go along the line $x = 0$ then the limit will be zero. The limit does not exist.

4. $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

   If we put this function in polar coordinates we get $f(r, \theta) = r \cos\theta \sin\theta$, which approaches zero as $r \to 0$.

   Or we could suppose without loss of generality that $y \geq x$ at a given point (x,y) (since the function is symmetric in the two variables). Then $|f(x, y)| \leq |y^2/\sqrt{y^2}| = |y|$, which goes to zero as y does. Thus the limit is zero.

Determine the set of points on which the function is continuous:

1. $G(x, y) = \ln(x^2 + y^2 - 4)$

   Continuous as long as $x^2 + y^2 - 4 > 0$, or $x^2 + y^2 > 4$.

2. $f(x, y, z) = \arcsin(x^2 + y^2 + z^2)$

   Continuous as long as $-1 \leq x^2 + y^2 + z^2 \leq 1$, which is the same as the unit sphere.

3. $f(x, y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & (x, y) \neq (0, 0) \\ 1 & (x, y) = (0, 0) \end{cases}$

   Continuous everywhere except maybe at $(0, 0)$, so we have to check whether the limit exists at that point. The limit as $(x, y) \to 0$ is zero, so the function is not continuous at the origin.