13.4: Velocity and Acceleration

Friday, February 19

Some Formulas

- $v = \mathbf{v}$
- $\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$
- $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$

More Quadratic Surfaces

Classify and sketch the surface:

1. $x^2 + z^2 - 2z = y^2$: hyperboloid, 1 sheet, skewered by y-axis.
2. $x^2+2x+y^2 = z^2-4z+1$: hyperboloid, 2 sheets, skewered by z-axis.
3. $x^2 + y^2 - 2y = 3 - z^2$: ellipsoid.
4. $y^2 + 4y + x^2 - 2x + 5 = -z^2$: the single point (1,-2,0)
5. $x - y^2 = 3 - z^2$: hypberbolic paraboloid (saddle)
6. $y - x^2 = 2 + z^2$: elliptic paraboloid (bowl)

Arc Length

The position vector of a child going down a slide is given (in meters) by $\mathbf{r}(t) = \langle \sin t, \cos t, 5 - t\rangle$. How long is the slide? Where is the child after sliding 3 meters?

$$\begin{aligned} L &= \int |d\mathbf{r}| \\ &= \int \sqrt{(\sin'(t))^2 + (\cos'(t))^2 + ((5-t)')^2}\, dt \\ &= \int \sqrt{2}\, dt \\ &= 5\sqrt{2}. \end{aligned}$$

This means that $L(t) = \sqrt{2}t$, so $t = L/\sqrt{2}$. After sliding 3 meters, the child's position is $\langle \arcsin(3/\sqrt{2}), \arccos(3/\sqrt{2}), 5 - 3/\sqrt{2}\rangle$.

Return of Projectile Motion

A projectile is fired at angle α with initial velocity v_0. (1) If the downward acceleration due to gravity is $-g$, express the projectile's position and velocity as a function of time. (2) Describe and sketch the particle's trajectory. (3) What is the projectile's velocity at the time it hits the ground? (4) Find the point (in time and space) when the particle's velocity is at a minimum. Sketch the velocity and acceleration vectors at this point.

1. Velocity is $\langle v_0 \cos\alpha, v_0 \sin\alpha - gt\rangle$. Position is $\langle v_0 \cos(\alpha)t, v_0 \sin(\alpha)t - \frac{1}{2}gt^2\rangle$.
2. Assuming $0 < \alpha < \pi/2$, the trajectory is a parabola.
3. The velocity at the time the projectile hits the ground is v_0, at time $t = \frac{2}{g}v_0 \sin\alpha$.
4. The particle's velocity is at a minimum at the top of its arc, when $t = \frac{v_0}{g}\sin\alpha$. The velocity vector is horizontal and so perpendicular to the acceleration vector.

Uniform Circular Motion

- Find the curvature of a circle of radius r.

  Take $\kappa = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$, and say the particle moves at constant angular speed 1. Then its position vector is $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$ and so $\mathbf{r}'(t) = r\langle -\sin t, \cos t\rangle$. This means that $\mathbf{T}(t) = \langle -\sin t, \cos t\rangle$, and $|\mathbf{T}'(t)| = |\langle -\cos t, -\sin t| = 1$. The curvature is therefore $1/r$.

- A parcticle moves with position $\mathbf{r}(t)$, velocity $\mathbf{v}(t)$, and acceleration $\mathbf{a}(t)$. If $|\mathbf{r}(t)|$ is constant, show that $\mathbf{r}\cdot\mathbf{v} = 0$ at all points in time. If $|\mathbf{v}(t)|$ is constant, show that $\mathbf{v}\cdot\mathbf{a} = 0$ at all points in time.

  If $\mathbf{r}(t)$ is a constant c, then $x(t)^2 + y(t)^2 + z(t)^2 = c^2$. Taking derivatives on both sides gives $0 = 2xx' + 2yy' + 2zz' = 2\mathbf{r}\cdot\mathbf{v}$. The proof that velocity and acceleration are perpendicular is the same.

- An object is traveling in a circle with constant speed. Find the relation between the acceleration and velocity of the particle and the radius of the circle.

  Use the formula that $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$. Since $\mathbf{a}\cdot\mathbf{v} = 0, \mathbf{T} = 0$. The curvature of the circle is a constant $1/r$, so $|a| = \frac{v^2}{r}$.