13.1-13.2: Vector Functions

Friday, February 12

Warmup

Critique the given answer to the following problem:

1. Do the lines given by $L_1 = \langle 1, 2, 3\rangle + t\langle 2, -1, -2\rangle$ and $L_2 = \langle 3, 6, -1\rangle + t\langle -1, 3, 0\rangle$ intersect?

   Check whether the lines intersect by seeing whether there is a value of t such that the two points given by t are equal:

$$\langle 1, 2, 3\rangle + t\langle 2, -1, -2\rangle = \langle 3, 6, -1\rangle + t\langle -1, 3, 0\rangle$$
$$1 + 2t = 3 - t$$
$$2 - t = 6 + 3t$$
$$3 - 2t = -1.$$

   The first equation gives $t = 2/3$, the second gives $t = -1$, and the third gives $t = 2$. There is no value of t that satisfies all three equations, and the lines therefore do not intersect.

The issue is using the same value of "t" for both equations simultaneously. If the parametric equations described particles traveling on the line over time, then this answer would show that the particles do not collide, but does not show that their paths do not cross. In fact, the lines do intersect at the point $(5, 0, -1)$.

Sketch the curves given by the following equations:

1. $z = y^2 - x$: A trough with parabolic walls, and floor sliding downward as x increases (like a rain gutter, perhaps).
2. $z^2 = x^2$: Two planes that intersect at $90°$ angles at the line $x = z = 0$.
3. $x^2 = z + y^2 + 1$: $(z + 1) = x^2 - y^2$, so it's a hyperbolic paraboloid centered at the point $(0, 0, -1)$.
4. $z = \sin t$, $x = \cos t$, $y = 1$: A circle.

Space Curves

At what points does the curve $\mathbf{r}(t) = t\mathbf{i} + (2t - t^2)\mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?
$\mathbf{j} = 0$ for all times t, so the points on the curve must satisfy the relation $z = x^2$. This means that $(2t - t^2) = t^2$, which has solutions $t = 0$ and $t = 1$. The curve intersects the paraboloid at the points $(0, 0, 0)$ and $(1, 0, 1)$.

Derivatives and Integrals

A particle is moving along the curve $x = t, y = t^2 - t$.

1. Find the unit vector tangent to its path at $t = 2$. The tangent vector is $\langle 1, 2t-1\rangle = \langle 1, 3\rangle$.

2. Find its speed at $t = 2$. $|T(t)| = \sqrt{1^2 + 3^2} = \sqrt{10}$.

3. At what points in time is the particle moving directly away from the origin?

   We would need its position vector $\langle t, t^2 - t\rangle$ and its velocity vector $\langle 1, 2t-1\rangle$ to be parallel, so $k\langle 1, 2t-1\rangle = \langle t, t^2 - t\rangle$ for some values of k and t. The only solution is $t = 0$, when the particle is at the origin.

4. When and where is the particle's speed at a minimum?

   $|T(t)|^2 = T(t) \cdot T(t) = 1 + (2t-1)^2$, which hits a minimum of 1 at $t = 1/2$. The particle is then at the point $(1/2, -1/4)$.

Find a vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point $(3, 4, 2)$.
Finding parametric equations is a little unpleasant, so handle the vectors one pair at a time using implicit differentiation: $2x + 2yy' = 0$, so $dx/dy = -y/x = -4/3$. Similarly, $dz/dy = -y/z = -2$. So one tangent vector is $\langle -4/3, 1, -2\rangle$, and the tangent line can be given by $\langle 3, 4, 2\rangle + t\langle -4/3, 1, -2\rangle$.

The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle$ and $\mathbf{r}_2(t) = \langle \sin t \sin 2t, t\rangle$ intersect at the origin. Explain how to find their angle of intersection.
Find the tangent vectors: $\langle 1, 2t, 3t^2\rangle = \langle 1, 0, 0\rangle$ and $\langle \cos t, 2\cos 2t, 1\rangle = \langle 1, 2, 1\rangle$. Then take the dot product.

Bonus: Graph the curve with parametric equations $x = \cos t, y = \sin t \cos t, z = \sin^2 t$. It may help to graph the projections of the curve on the x-y, x-z, and y-z planes.
Have fun! The particle always lies on the surface of the unit sphere.