12.4-5: Cross Product, Lines and Planes

Friday, February 5

Warmup

1. Find the projection of the vector $\langle 1, 2\rangle$ onto $\langle 2, -1\rangle$. Make a sketch.

   The projection is 0 (or $\langle 0, 0\rangle$), since the two vectors are orthogonal.

2. Find the projection of the vector $\langle 3, 2, 1\rangle$ onto $\langle 2, 0, 0\rangle$. Make a sketch.

   This is the same as the projection onto the x-axis, so the projection is $\langle 3, 0, 0\rangle$.

3. One student is pushing a door open with a force of 50N at a 45° angle relative to the body of the door. Another is pushing it shut with a force of 40N perpendicular to the body of the door. A third is pushing with a force of 30N along the axis from the end of the door to its hinge. Which way will the door move? Make a sketch.

   Inward, since $40 > 50 \sin 45$.

4. Find the determinant of the matrix

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -2 & 0 \\ 0 & 3 & 3 \end{pmatrix}.$$

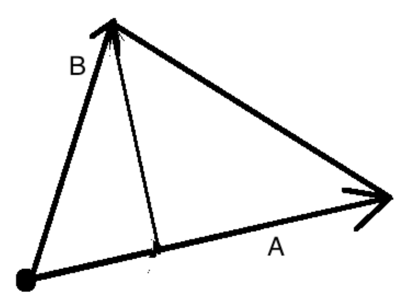
   What does it tell you about the volume of a parallelepiped with one vertex at the origin and adjacent vertices at $(0, 3, 3), (1, -2, 0)$, and $(-1, -1, 1)$?

   The determinant is $(-1)(-2)(3) + (-1)(0)(0) + (1)(1)(3) - (-1)(0)(3) - (-1)(1)(3) - (1)(-2)(0) = 6 + 3 + 3 = 12$. This is the triple product of the three vectors representing adjacent edges of the parallelepiped, so the volume of the parallelepiped is 12.

Matching

What do each of the following expressions have to say about the picture below? (Suppose the picture is embedded in three-dimensional space, so the cross product is well-defined.)

1. $a \times b$: a vector pointing out of the page.
2. $b \times a$: a vector pointing into the page.
3. $(a \cdot b)/|a|$: the base length of the right triangle with b as its hypotenuse.
4. $a(a \cdot b)/|a|^2$: The vector representing the base of said triangle.
5. $b - a(a \cdot b)/|a|^2$: The vector representing the height of said triangle.
6. $|a \times b|/|a|$: The height of the above triangle.
7. $|a \times b|/2$: The area of the whole (larger) triangle.



Lines and Planes

1. If a plane is represented in the form $ax + by + cz = d$, what is the relation between $\langle a, b, c\rangle$ and the plane?

   It's normal to the plane.

2. If the plane can also be written in the form $\{P_0 + t\mathbf{u} + s\mathbf{v} : s, t \in \mathbb{R}\}$, what is the relation between $\mathbf{u}, \mathbf{v}$, and $\langle a, b, c\rangle$?

   $\langle a, b, c\rangle$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, so the cross product of $\mathbf{u}$ and $\mathbf{v}$ is parallel to $\langle a, b, c\rangle$.

3. How would you describe the set of points $\mathbf{x}$ such that $\mathbf{a} \cdot \mathbf{x} = 1$ for some nonzero vector $\mathbf{a}$ in $\mathbb{R}^2$? $\mathbb{R}^3$? $\mathbb{R}^4$??

   In 2D: a line. 3D: a plane. 4D: a hyperplane.

Many Formulas

1. Given two parallel lines of the form $\mathbf{u_0} + t\mathbf{v}$ and $\mathbf{v_0} + t\mathbf{v}$, find a formula for the distance between the two lines. (Hint: it's the height of a triangle).

   Try $|(U_0 - V_0) \times v|/|v|$, using one of the expressions from the previous page.

2. Find a formula for the distance between two planes in the form $\mathbf{n} \cdot x = k_1$ and $\mathbf{n} \cdot x = k_2$.

   The shortest route between the planes is along the vector $\mathbf{n}$, so if we start at a point u_0, then our destination point is of the form $u_0 + t\mathbf{n}$. This means that

$$\begin{aligned} k_2 &= \mathbf{n} \cdot (u_0 + t\mathbf{n}) \\ &= \mathbf{n} \cdot u_0 + t\mathbf{n} \cdot \mathbf{n} \\ &= k_1 + t|\mathbf{n}|^2 \\ t &= (k_2 - k_1)/|n|^2 \end{aligned}$$

   So the distance between the two points is equal to $|t\mathbf{n}|$, which is equal to $(k_2 - k_1)/|n|$.

3. How can you use triple products to tell whether two lines of the form $\mathbf{u_0} + t\mathbf{u}$ and $\mathbf{v_0} + t\mathbf{v}$ are skew lines?

   Pick four points on the lines, treat them as vertices of a parallelepiped, and check whether the triple product is zero. If it is, the parallelepiped has zero volume and so all four points lie on a plane. If not, then the lines are skew lines.

   One such expression that works is

$$\begin{vmatrix} u_0 - v_0 \\ u \\ v \end{vmatrix}$$

4. Come up with some questions of your own!