# 12.2: Vectors

Friday, January 29

## Warmup

1. $\lim_{x\to 0} \frac{\sin x}{x} = \frac{\cos(0)}{1} = 1$

2. $\lim_{x\to 0} \frac{1-\cos x}{x} = \frac{\sin(0)}{1} = 0$

3. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim \frac{\sin x}{2x} = 1/2$

4. If $y(t) = t^2$ and $x(t) = 1 - \cos t$, what is the slope when $t = 0$?

$$\lim_{x\to 0} \frac{y'(t)}{x'(t)} = \lim_{x\to 0} \frac{2t}{\sin t} = 2.$$

5. Alice and Bob find a treasure map, which gives the following instructions:

*Go 2 miles north. Go 3 miles east. Go 4 miles west. Go 1 mile south. Go 2 miles west.*

(a) How far away from their starting point will they end up?
If $(1, 0)$ is east and $(0, 1)$ is north, they will end up at $(-2, 1)$, so $\sqrt{5}$ miles away.

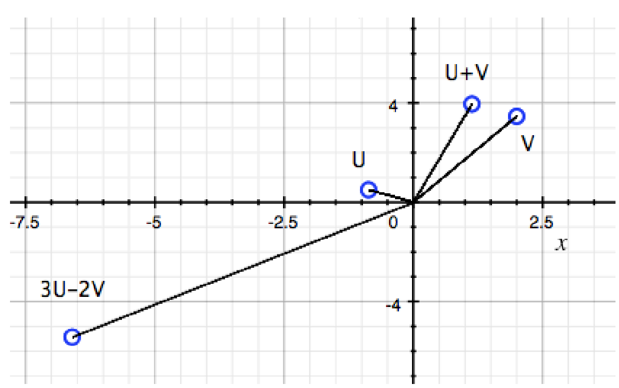
(b) If they start at the right spot but accidentally follow the instructions in reverse order, how far from the treasure will they end up?
It doesn't matter what order they carry out the directions (assuming there are no obstacles in their path)...they will end up exactly at the treasure. Lesson: you can add vectors in any order

## Vectors!

1. Plot the vector $v = \langle 2, 2\sqrt{3}\rangle$. What angle does it make with the positive x-axis? **Note: these numbers are different from the ones presented in class, but the procedure for solving the problems is largely the same.**

   Plots below. $v$ makes an angle of $\pi/3$ with the positive x-axis.

2. Find a unit vector $u$ that is perpendicular to $v$, and plot $u$, $u + v$, and $3u - 2v$ relative to the origin.
   $u = \langle -\sqrt{3}/2, 1/2\rangle$ will do. The other option is $u = \langle \sqrt{3}/2, -1/2\rangle$.

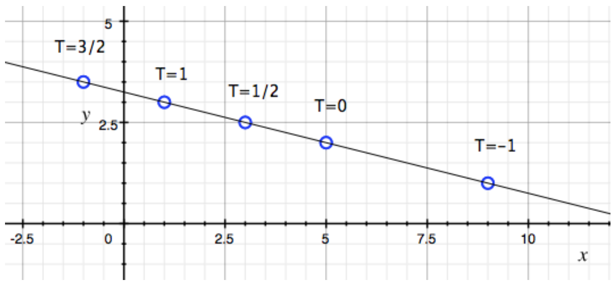



3. Find $|v|$, $|u + v|$, and $|3u - 2v|$. How is the Pythagorean Theorem relevant?

$|v| = 4$, $|u+v| = \sqrt{17}$, $|3u-2v| = \sqrt{35}$. Pythagorean Theorem for vectors: if $u$ and $v$ are perpendicular then $|u+v|^2 = |u|^2 + |v|^2$.

4. Plot the vectors $w = \langle 1, 3\rangle$ and $z = \langle 5, 2\rangle$. Plot the vectors $tw + (1-t)z$ for $t = -1, 0, 1/2, 3/2$. Make observations.

   The endpoints of all of these vectors lie on a line.

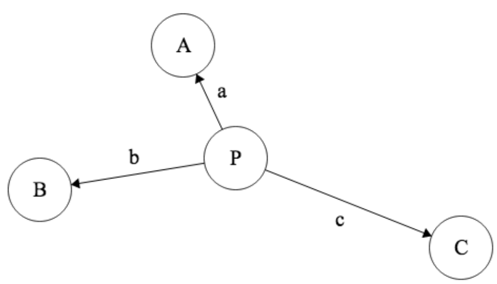



## More Vectors!

1. A box is neither moving nor accelerating (relative to the table it sits on). What is the net force on the box?

   Zero.

2. The three forces $a$, $b$, and $c$ are acting on the object at $P$, and the net force is zero.




   Treating $P = \vec{OP}$, $A = \vec{OA}$, $B = \vec{OB}$, and $C = \vec{OC}$ as vectors starting from the origin (not pictured), express $a, , b, c$ in terms of $P, A, B, C$.

$$a = A - P, b = B - P, c = C - P.$$

3. Given that the net force is zero, express $P$ in terms of $A$, $B$, and $C$.

   $0 = a + b + c = (A - P) + (B - P) + (C - P) = (A + B + C) - 3P$, so $P = (A + B + C)/3$. Another view: if A, B, C are point masses of equal weight then $P$ is the center of gravity of those masses.

4. A boatman wants to cross a canal that is 2km wide and wants to land on a point 4km upstream from his starting point. The current in the canal flows at 2km/h and the speed of his boat is 13km/h. **Note: These numbers have been changed to make the answer nicer.**

(a) In what direction should he steer?

Try solving for time first: let the boatman start at the point $(0, 0)$, with destination $(2, 2)$, and assume he travels in the correct direction for time $t$. He will have crossed the 2km canal 4km upstream, and the river will have pushed him back $(2t)$km. Drawing a right triangle, this means that $2^2 + (4 + 2t)^2 = 13t^2$. This leads to the quadratic equation $165t^2 - 16t - 20 = 0$, which has the positive solution $t = 2/5$ of an hour, or 24 minutes. The optimal direction is therefore $\theta = \arctan(12/5) \approx 67°$.

(b) How long will the trip take?

See above.