16.7-8: Surface Integrals, Stokes' Theorem

Friday, April 29

Generic Surface Integral

(16.7.23) Find the surface integral $\iint_S \mathbf{F} \cdot dS$ where $\mathbf{F}(x, y, z) = \langle xy, yz, zx \rangle$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ lying above the square $0 \leq x, y \leq 1$ and has upward orientation.

S is the graph of a function in x and y, so Formula 9 from the chapter applies here:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \\
&= \iint_D (-(xy)(-2x) - yz(-2y) + zx)\, dA \\
&= \iint_D (2x^2y + 2y^2z + zx)\, dA \\
&= \iint_D (2x^2y + 2y^2(4 - x^2 - y^2) + (4 - x^2 - y^2)x)\, dA \\
&= \int_{y=0}^1 \int_{x=0}^1 (2x^2y + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - y^2x)\, dx\, dy \\
&= \int_{y=0}^1 \frac{2}{3}y + 8y^2 - \frac{2}{3}y^2 - 2y^4 + 2 - \frac{1}{4} - y^2/2\, dy \\
&= 1/3 + 8/3 - 2/9 - 2/5 + 2 - 1/4 - 1/6 \\
&= 3 + 173/180.
\end{aligned}$$

Special Surface Integral

(16.7.49) An electric charge at the origin generates an electric field given by $\mathbf{E}(r, \theta, \phi) = \frac{cr}{|r|^3}$, where c is a constant. Show that if S is the surface of a sphere centered at the origin then $\iint_S \mathbf{E} \cdot d\mathbf{S}$ does not depend on the radius of the sphere. What does this mean?

$\mathbf{E}$ is always parallel to the vector normal to the surface of the sphere, so $\mathbf{E} \cdot \mathbf{N} = \frac{c}{r^2}$, a constant. Therefore $\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_S \frac{c}{r^2}\, dS = 4\pi c$, where we used the fact that r is constant on S.

Stokes' Theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

(16.8.7) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle x + y^2, y + z^2, z + x^2 \rangle$ and C is the (counterclockwise-oriented) boundary of the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

The triangle is defined by the plane $x + y + z = 1$ and so has constant normal vector $\langle 1, 1, 1 \rangle/\sqrt{3}$. Therefore,

$$
\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{N}\,dS \\
&= \iint_S \langle -2z,-2x,-2y\rangle\cdot\langle 1,1,1\rangle/\sqrt{3}\,dS \\
&= -2/\sqrt{3}\iint_S (x+y+z)\,dS \\
&= -2/\sqrt{3}\iint_S dS,
\end{aligned}
$$

so we just need to find the area of the triangle. It's equilateral, so we can do this geometrically, getting that the area is $\sqrt{3}/2$. The integral is therefore -1.

True or False?

1. If $\mathbf{F}$ is a vector field then $\nabla\cdot\mathbf{F}$ is a vector field. FALSE: the divergence is a scalar function.
2. If $\mathbf{F}$ is a vector field then $\nabla\times\mathbf{F}$ is a vector field. TRUE.
3. I f has continuous partial derivatives on $\mathbb{R}^3$ then $\nabla\cdot(\nabla\times f)=0$. TRUE.
4. If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle then $\int_C \nabla f\cdot d\mathbf{r}=0$. FALSE: this is necessarily true only if the curl of f is zero.
5. If $\mathbf{F}=\langle P,Q\rangle$ and $P_y=Q_x$ in an open region D then $\mathbf{F}$ is conservative. FALSE: D must be simply connected.
6. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields and $\nabla\times\mathbf{F}=\nabla\times\mathbf{G}$ then $\mathbf{F}=\mathbf{G}$. FALSE: F can be G plus any function whose curl is zero.
7. The work done by a conservative force field in moving a particle around a closed path is zero. TRUE.
8. There is a vector field $\mathbf{F}$ such that $\nabla\times\mathbf{F}\langle x,y,z\rangle$. FALSE: this function has non-zero divergence, but an earlier true/false implies that the divergence of the curl of any smooth function is zero.