16.3-5: Green's Theorem, Curl

Friday, April 22

Work

(16.3.23) Find the work done by the force field $\mathbf{F}(x,y) = \langle 2y^{3/2}, 3x\sqrt{y}\rangle$ in moving an object from $(1,1)$ to $(2,4)$.

As $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the field is conservative. Integrating $2y^{3/2}$ over x gives $2xy^{3/2} + g(y)$ and integrating $3x\sqrt{y}$ over y gives $2xy^{3/2} + h(x)$, so $\mathbf{F} = \nabla f$ where f is of the form $f(x,y) = 2xy^{3/2} + K$. Then the work done is $f(2,4) - f(1,1) = 30$.

Green's Theorem

(16.4.21) If C is the line segment from (x_1,y_1) to (x_2,y_2), show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1.$$

Use this to find a formula for the area of a triangle with vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$.

Parametrize the line (in vector form) as $r(t) = A + t(B-A)$, where A and B are the start and end points. If we do this with $A = (x_1,y_1)$ and $B = (x_2,y_2)$ we get $r(t) = \langle x_1 + t(x_2-x_1), y_1 + t(y_2-y_1)\rangle$ and $r'(t) = \langle x_2 - x_1, y_2 - y_1\rangle$. Therefore,

$$\begin{aligned}
\int_C x\,dy - y\,dx &= \int_{t=0}^{1} [x_1 + t(x_2-x_1)][y_2-y_1] - [y_1 + t(y_2-y_1)][x_2-x_1]\,dt \\
&= (y_2-y_1)(tx_1 + \frac{t^2}{2}(x_2-x_1)) - (x_2-x_1)(ty_1 + \frac{t^2}{2}(y_2-y_1))|_{t=0}^{1} \\
&= (y_2-y_1)(x_1+x_2)/2 + (x_1-x_2)(y_1+y_2)/2 \\
&= x_1y_2 - x_2y_1.
\end{aligned}$$

Then with $P = -y, Q = x$, we get with Green's Theorem that

$$\begin{aligned}
\int_C x\,dy - y\,dx &= \int_C P\,dx + Q\,dy \\
&= \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA \\
&= \iint_A 2\,dA \\
&= 2A,
\end{aligned}$$

where A is the area of the triangle. So assuming $(x_1,y_1) \to (x_2,y_2) \to (x_3,y_3)$ is traveling in the counter-clockwise direction, we conclude that

$$A = \frac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3].$$

Curl

Use the curl operator to determine whether the vector field $\mathbf{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ is conservative.

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\rangle \\ &= \langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3\rangle \\ &= \langle 0, 0, 0\rangle.\end{aligned}$$

The field is conservative.

Divergence

A charged particle at the origin generates the electric field $\mathbf{E}(x, y) = \langle x/(x^2 + y^2)^{3/2}, y/(x^2 + y^2)^{3/2}\rangle$.

1. Find $\nabla \cdot \mathbf{E}$.
   $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{-1}{(x^2+y^2)^{3/2}}$.
2. Find curves C_1 and C_2 such that $\oint_{C_1} \mathbf{E} \cdot \mathbf{n}\, ds = 0$ and $\oint_{C_2} \mathbf{E} \cdot \mathbf{n}\, ds \neq 0$.

   So. . . this didn't turn out quite the way I expected. I had thought the divergence would be zero and so any closed loop not containing the origin would have surface integral zero, but as it turns out Coulomb's law (and the law of gravity) imply zero divergence only when viewed in **three** dimensions, not two.

   Anyhow, most integrals here will be non-zero because the divergence is always negative. If you integrate over a "curve" that encloses zero area then you will get a surface integral of zero.

True/False

1. If $\mathbf{F}$ is conservative then $\nabla \times \mathbf{F} = 0$: TRUE
2. If $\mathbf{F}$ is conservative then $\nabla \cdot \mathbf{F} = 0$: FALSE. This would imply that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ (or analogously in any number of dimensions), but lots of functions (e.g. $f(x, y) = x^2 + y^2$) violate this equality.
3. If $\nabla \times \mathbf{F} = 0$ then $\mathbf{F}$ is conservative: FALSE. This was true as long as $\mathbf{F}$ is defined on all of $\mathbb{R}^3$.
4. Green's Theorem is just the Divergence Theorem in two dimensions. FALSE: it's Stokes' Theorem in two dimensions.
5. curl(div($\mathbf{F}$)) is not a meaningful expression. TRUE, since curl must take a 3-D vector field as its argument, but div(F) is a scalar.