16.2-3: Line Integrals

Friday, April 15

Recap: Change of Coordinates

(15.9.15) Evaluate the integral $\iint_R (x-3y)\,dA$ where R is the triangular region with vertices $(0,0), (2,1), (1,2)$ given the transformation $x = 2u + v$, $y = u + 2v$.
Solving for u and v in terms of x and y gives $u = (2x - y)/3$, $v = (2y - x)/3$. This change of coordinates moves the three vertices of the triangle to $(0,0)$, $(1,0)$, and $(0,1)$, and $\partial(x,y)/\partial(u,v) = 2 \cdot 2 - 1 \cdot 1 = 3$ (note that the area of the original triangle is 3 times the area of the transformed one), so the integral becomes

$$\begin{aligned}
\iint_R (x-3y)\,dA &= \int_{u=0}^{1}\int_{v=0}^{1-u} (2u + v - 3(u+2v))[3]\,du\,dv \\
&= 3\int_{u=0}^{1}\int_{v=0}^{1-u} -u - 5v\,du\,dv \\
&= -3\int_{u=0}^{1} [uv + \frac{5}{2}v^2]_0^{1-u}\,du \\
&= -3\int_{u=0}^{1} u(1-u) + \frac{5}{2}(u^2 - 2u + 1)\,du \\
&= -3\int_{u=0}^{1} \frac{3}{2}u^2 - 4u + \frac{5}{2}\,du \\
&= -3[\frac{1}{2}u^3 - 2u^2 + \frac{5}{2}u]_0^1 \\
&= -3.
\end{aligned}$$

Line Integrals

Find the work done by the force field $\mathbf{F}(x,y) = \langle x - y^2, y - x^2 \rangle$ on a praticle that moves along the line segment from $(0,0)$ to $(2,1)$.
Define a vector function $\mathbf{r}(t)$ so that $\mathbf{r}(0) = (0,0)$ and $\mathbf{r}(1) = (2,1)$: $\mathbf{r}(t) = (2t, t)$. Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_{t=0}^{1} \langle 2t - t^2, t - 4t^2 \rangle \cdot \langle 2, 1 \rangle\,dt \\
&= \int_{t=0}^{1} 5t - 6t^2\,dt \\
&= [\frac{5}{2}t^2 - 2t^3]_0^1 \\
&= \frac{1}{2}.
\end{aligned}$$

The net work is positive.

A student swings a ball of mass m on a string of radius r in a vertical circle. Use a line integral to calculate the work that gravity does on the ball (given constant downward force mg)...

1. as the ball goes from the top of its arc to the bottom.

   Parametrize the curve by $\mathbf{r}(t) = \langle r\cos t, r\sin t\rangle$, and get

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{t=\pi/2}^{3\pi/2} \langle 0, -mg\rangle \cdot \langle -r\sin t, r\cos t\rangle\, dt \\
&= -mgr \int_{t=\pi/2}^{3\pi/2} \cos t\, dt \\
&= -mgr \sin t\Big|_{t=\pi/2}^{3\pi/2} \\
&= 2mgr.
\end{aligned}$$

   Alternately, the field is conservative since the force is the gradient of $E(x, y) = -mgy$, so the total work is $-mg(-r) - mg(r) = 2mgr$.

2. over one complete revolution.

   Same setup as before, but since t goes from 0 to 2π (or some equivalent) the net work will be zero. Also, because the field is conservative the integral over any closed curve is zero.

True or False?

1. The integral $\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{1} \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi$ gives the volume of 1/4 of a sphere. FALSE for two reasons: the $\sin\theta$ should be $\sin\phi$, and even with this change it only gives the volume of 1/8 of a sphere.

2. $\int_{r=-1}^{1}\int_{\theta=0}^{1} e^{r^2+\theta^2}\, d\theta\, dr = \left[\int_{r=-1}^{1} e^{r^2}\, dr\right]\left[\int_{\theta=0}^{1} e^{\theta^2}\, d\theta\right]$

   TRUE.

3. If C is a closed curve then $\int_C f\, ds = 0$ for any function f.

   Very FALSE.

4. If $\int_C f\, ds = 0$ then C is a closed curve. FALSE: for example, the integral of $\sin x$ as x goes from 0 to 2π.

5. If the work done by a force $\mathbf{F}$ on an object moving along a curve is W, then if the object moves along the curve in the opposite direction the work done by $\mathbf{F}$ will be $-W$. TRUE.

6. If a particle moves along a curve C, the total work done by a force $\mathbf{F}$ on the object is independent of how quickly the particle moves. TRUE.

7. If a force points only in the x direction then the work done by the force on a particle depends only on the particle's starting and ending x-positions. FALSE: the force's strength could depend on the y-position of the particle (Imagine swimming up a river versus walking along the bank. The river's current does different amounts of work.)