15.5-8: Surface Area, Triple Integrals

Friday, April 8

Surface Area

Using the formula $A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$, find the surface area of a sphere of radius a. (Hint: after finding f_x and f_y, convert to polar coordinates)
Say $f(x, y) = \sqrt{a^2 - x^2 - y^2}$. Then $f_x(x, y) = \frac{-x}{f(x,y)}$ and $f_y = \frac{-y}{f(x,y)}$, so $f_x^2 + f_y^2 = -(x^2 + y^2)/f(x, y)^2$. Multiplying the integral by 2 to account for both halves of the sphere and converting to polar coordinates gives

$$\begin{aligned}
A(S) &= 2 \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \sqrt{1 - \frac{-r^2}{a^2 - r^2}} r\, dr\, d\theta\, dr \\
&= 4\pi \int_{r=0}^{a} r \sqrt{\frac{1}{a^2 - r^2}}\, dr \\
&= 4\pi \int_{r=0}^{a} r(a^2 - r^2)^{-1/2}\, dr \\
&= 4\pi[-\sqrt{a^2 - r^2}]_0^a \\
&= 4\pi(0 + a^2) \\
&= 4\pi a^2.
\end{aligned}$$

Triple Integrals

Sketch the region E bounded by the surfaces $y = x^2, z = 0, y+2z = 4$. Express the integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral in the order of your choice, and find the volume of the region.

One potential order to find the volume:

$$\begin{aligned}
V &= \iiint dV \\
&= \int_{x=-2}^{2} \int_{y=0}^{x^2} \int_{z=0}^{(4-y)/2} dz\, dy\, dz \\
&= \int_{x=-2}^{2} \int_{y=0}^{x^2} (4 - y)/2\, dy\, dx \\
&= \int_{x=-2}^{2} [4y - y^2/4]_0^{x^2}\, dx \\
&= \int_{x=-2}^{2} 4x^2 - x^4/4\, dx \\
&= [\frac{4}{3}x^3 - \frac{1}{20}x^5]_{-2}^2 \\
&= \frac{64}{3} - \frac{16}{5}.
\end{aligned}$$

Cylindrical Coordinates

Find the volume of a cylinder using cylindrical coordinates. Set up the integral at least three different ways, and give a geometric interpretation of each ordering. Make lots of sketches.

If r is the variable in the outer integral, you summing up over cylindrical shells. If z is the outer variable, you are summing up disks. If θ is the outer variable, you are summing up triangular wedges.

Spherical Coordinates

Explain, with pictures, why $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.
Start at a point ρ, θ, ϕ, and change by small amounts $d\rho, d\theta, d\phi$. The volume covered by the difference will (approximately!) be a rectangular prism with side lengths $d\rho$, $\rho \, d\theta$, and $\rho \, d\phi$ (the latter two lengths are magnified by ρ) because changes in an angle make a bigger change in distance the farther the point is from the origin.

Why is longitude measured from $180°E$ to $180°W$ but latitude measured only from $90°N$ to $90°S$?
So that each point has a unique coordinate. If we tried to go to $180°N$, we would end up back on the equator (which is already specified by $0°N$).

Find the volume of a sphere using spherical coordinates. Set up the integral at least three different ways and give a geometric interpretation of each ordering. Make lots of sketches.

You will hopefully get $\frac{4}{3}\pi r^3$ for a sphere with radius r. If you integrade over ρ last, you will be summing over spherical shells (each of which has surface area $4\pi\rho^2$!), if you integrate over ϕ last you will be summing up conical-paper-cup-shaped wedges, and if you integrate over θ last you will be summing up orange-peel-shaped wedges.